FIELD

A non-empty set F with 2 binary composition denoted by '+' & '.' is called a field if the following condition are satisfied.

A (i) Closure: $a, b \in F \Rightarrow a+b \in F$

(ii) Associative: $a+(b+c) = (a+b)+c \quad \forall a, b, c \in F$

(iii) Additive identity: $\exists$ an element $0 \in F$ such that
$$a+0 = a = 0+a \quad \forall a \in F$$

(iv) Additive inverse: for each $a \in F$, $\exists$ $b \in F$ such that
$$a+b = 0 = b+a$$
then b is called additive inverse ~~of~~ of a and is also written in the the form $b = -a$.

(v) Commutative: $a+b = b+a \quad \forall a, b \in F$

M (i) Closure: $a, b \in F \Rightarrow ab \in F$

(ii) Associative: $a(bc) = (ab)c \quad \forall a, b, c \in F$

(iii) Multiplicative identity: $\exists\ 1 \in F$ such that
$$a \cdot 1 = a = 1 \cdot a \quad \forall a \in F$$

(iv) Multiplicative inverse: for each $a \in F$, $\exists$ $b \in F$ such that
$$ab = 1 = ba$$
here b is called inverse of a & written as $b = a^{-1}$
$$\text{i.e. } aa^{-1} = 1 = a^{-1}a.$$

(v) Commutative: $ab = ba \quad \forall a, b \in F$

D (i) Left distributive law: $a(b+c) = ab + ac \quad \forall a, b, c \in F$

(ii) Right distributive law: $(a+b)c = ac + bc \quad \forall a, b, c \in F$

Note: F is a non-empty set, $\langle F, +, \cdot \rangle$ is a field if

(i) $\langle F, + \rangle$ is an abelian group.

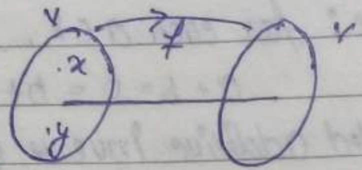
(ii) $\langle F, \cdot \rangle$ is an abelian group.

(iii) Distributive law holds.

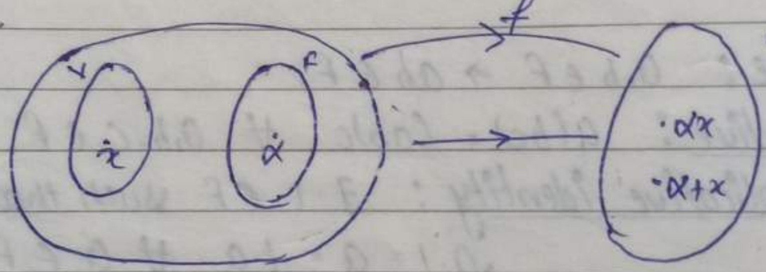
External And Internal Binary Compositions.

Let V & F be 2 non-empty set.

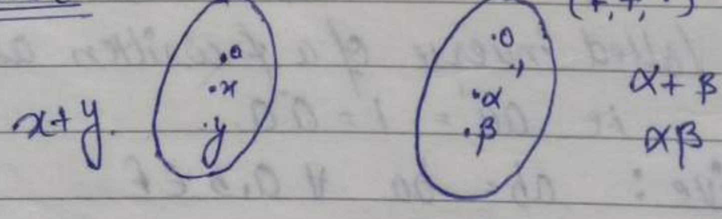
(i) A mapping $f: V \times V \to V$ is called an internal binary composition



(ii) A mapping $f: V \times F \to V$ is called external binary composition.



Vector Space



Let $\langle F, +, \cdot \rangle$ is a field then a non-empty set V together with 2 binary composition vector addition (internal composition) and scalar multiplicative (external composition) is called a vector space over the field F if the following condition is satisfied.

(A) $\langle V, + \rangle$ is an abelian group.

* (B) Any $u \in V$, $\alpha \in F \Rightarrow \alpha u \in V$, then the following condition are satisfied.

(i) $a(u+v) = au + av$
(ii) $(a+b)u = au + bu$
(iii) $a(bu) = (ab)u$

$\forall\ u, v \in V$ & $a, b \in F$

(iv) for the unit element $1 \in F$

$$1 \cdot u = u \ \forall\ u \in V$$

It is denoted by $V(F)$ [Vector space over the field F)

<u>Note</u> (i) $\mathbb{R}(\mathbb{R})$ is a Vector space w.r.to usual additional & multiplication

(ii) $\mathbb{C}(\mathbb{R})$ is a Vector space w.r.to usual additional & multiplication

(iii) $\mathbb{C}(\mathbb{C})$ is a Vector space w.r.to usual additional & multiplication.

(iv) $\mathbb{R}(\mathbb{C})$ is not a " " " " " " "

For example:

$$a \in \mathbb{R},\ \alpha \in \mathbb{C} \Rightarrow \alpha a \notin \mathbb{R}$$

$\mathbb{R}$: $a = 2$ $\quad$ $\mathbb{C}$: $\alpha = 1+i$

$2 \in \mathbb{R}$
$1+i \in \mathbb{C}$

$$(1+i)2 = 2 + 2i \notin \mathbb{R}.$$

<u>Note</u>: Rational no. p/q,

$q \neq 0$ and p & q are Co-prime.

(v) $\mathbb{Q}$ is not a Vector space over $\mathbb{C}$

for example.

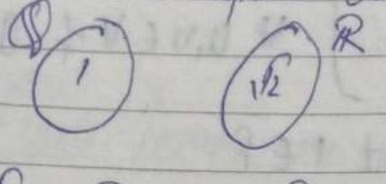
$\mathbb{Q}$: 1 $\quad$ $\mathbb{C}$: $\sqrt{2}$, $\sqrt{2}i$

$$1 \in \mathbb{Q},\ \sqrt{2} \in \mathbb{C} \Rightarrow \sqrt{2} \cdot 1 = \sqrt{2} \notin \mathbb{Q}$$

$1 \in \mathbb{Q}, \sqrt{2}i \in \mathbb{C} \Rightarrow 1 \cdot \sqrt{2}i = \sqrt{2}i \notin \mathbb{Q}$

$\therefore$ $\mathbb{Q}$ is not a Vector space over $\mathbb{C}$

(vi) $\mathbb{Q}$ is not a vector space over $\mathbb{R}$



$1 \in \mathbb{Q}, \sqrt{2} \in \mathbb{R} \Rightarrow 1 \cdot \sqrt{2} = \sqrt{2} \notin \mathbb{Q}$

$\therefore$ $\mathbb{Q}$ is not a Vector space over $\mathbb{R}$.

<u>Thm 1.</u> Elementary property of Vector space. Let $V(F)$ be a Vector space, let $u, v \in V$, $a \in F$ be arbitrary, then Prove that.

(i) $a \cdot 0 = 0$

(ii) $0u = 0$

(iii) $a(-u) = -(au) = (-a)u$

(iv) $a(u-v) = au - av$

(v) $au = 0,\ a \neq 0 \Rightarrow u = 0$.

<u>Proof</u>: (i) $0 + 0 = 0$

$a(0+0) = a \cdot 0$

$a \cdot 0 + a \cdot 0 = a0 + 0 \quad (\because a0 = a0 + 0)$

$a0 = 0$ (By left Cancellation law in $(V,+)$)

(ii) $0 + 0 = 0$

$(0+0)u = 0u$

$0u + 0u = 0u + 0 \quad (\because 0u = 0u + 0)$

$0u = 0$ (By left Cancellation law)

(iii) $0 = a \cdot 0$ (By i)

$0 = a(u + (-u)) \quad (\because u + (-u) = 0)$

au

Date

$0 = au + (a(-u))$

$\Rightarrow a(-u) = -(au)$ ($\because a(-u)$ is the additive inverse of (au))

(iv) $a(u-v) = a[u+(-v)]$

$= au + a(-v)$

$= au - av$ $\quad (\because a(-u) = -au)$

(v) $a \neq 0 \Rightarrow a^{-1}$ exist $\quad (\because a \in F)$

$aa^{-1} = 1$

$\because au = 0$

Premultiply by a^{-1}

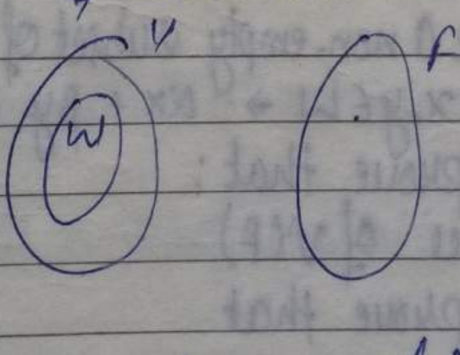
$a^{-1}(au) = a^{-1}0$

$(a^{-1}a)u = 0$

$1 \cdot u = 0$

$u = 0$

SUBSPACE: let $V(F)$ be the vector space a non-empty subset of W of V is called a subspace of V if W itself a vector space over F w.r.t. to composition of vector addition & scalar multiplication in V.



Note:

(i) $W \subseteq V$ is a subspace of V iff

(a) $x, y \in W \Rightarrow x+y \in W$

(b) $\alpha \in F, x \in W \Rightarrow \alpha x \in W$

(ii) W is subset of V is called a subspace of V iff

(a) $x, y \in W$ & $\alpha, \beta \in F$

$\Rightarrow \alpha x + \beta y \in W$.

Page No.
Date

Theorem: The necessary & sufficient Condition for non-empty Subset W of a Vector space $V(F)$ to be a subspace of V is $\alpha, \beta \in F$ & $x, y \in W \Rightarrow \alpha x + \beta y \in W$

Proof:

Let W be a subspace of $V(F)$ then
We have to prove that
$$\alpha, \beta \in F \ \& \ x, y \in W \Rightarrow x\alpha + \beta y \in W$$

Since,
W is a subspace of $V(F)$ then
By definition $W(F)$ is a vector space.
$\therefore \alpha \in F, x \in W \Rightarrow \alpha x \in W$
$\beta \in F, y \in W \Rightarrow \beta y \in W$ $\left\{ \begin{array}{l} W \text{ is a subspace} \\ \text{It is Vector Space.} \end{array} \right.$

$\therefore \alpha x \in W, \beta y \in W \Rightarrow \alpha x + \beta y \in W$ ($\because W$ is a Vector Space)

i.e $\alpha, \beta \in F$ & $x, y \in W$
$\Rightarrow \alpha x + \beta y \in W$

Conversely

Suppose that W be a non-empty subset of the Vector Space $V(F)$
s.t. $\alpha, \beta \in F$ & $x, y \in W \Rightarrow \alpha x + \beta y \in W$ —(i)
then We have to prove that:
W is a subspace of $V(F)$
i.e W have to prove that
(i) $x, y \in W \Rightarrow x + y \in W$
(ii) $\alpha \in F, x \in W \Rightarrow \alpha x \in W$

Let $\alpha = \beta = 1$ in eq^n (i)
$\therefore x, y \in W \Rightarrow x + y \in W$ ($\because F$ is field.)
— (A)

let $\beta = 0$

$\therefore \alpha \in F,\ x \in W \Rightarrow \alpha x \in W$ —— (B)

By (A) & (B)

$\therefore$ W is a subspace of $V(F)$

Theorem: The intersection of 2 subspaces W_1 & W_2 of a vector space $V(F)$ is also a subspace of $V(F)$.

Proof:

let W_1 & W_2 be 2 subspace of a vector space $V(F)$

then, we have to prove that $W_1 \cap W_2$ is a subspace of $V(F)$

i.e we have to prove that $\alpha, \beta \in F$ & $x, y \in W_1 \cap W_2$

$\Rightarrow \alpha x + \beta y \in W_1 \cap W_2$

Now, $x, y \in W_1 \cap W_2$ & $\alpha, \beta \in F$

$\Rightarrow x, y \in W_1$ & $x, y \in W_2$ also $\alpha, \beta \in F$

$\therefore W_1$ is a subspace of $V(F)$

$\therefore \alpha, \beta \in F$ & $x, y \in W_1 \Rightarrow \alpha x + \beta y \in W_1$

Also, W_2 is a subspace of $V(F)$

$\therefore \alpha, \beta \in F$ & $x, y \in W_2 \Rightarrow \alpha x + \beta y \in W_2$

Hence,

$\alpha, \beta \in F$ & $x, y \in W_1 \cap W_2 \Rightarrow \alpha x + \beta y \in W_1 \cap W_2$

$\therefore W_1 \cap W_2$ is a subspace of $V(F)$

Theorem 3. The union of two subspaces of a vector space $V(F)$ is not necessarily a subspace of $V(F)$

Proof:

let $W_1 = \{(a, 0, 0) : a \in \mathbb{R}\}$

$W_2 = \{(0, a, 0) : a \in \mathbb{R}\}$

let W_1 & W_2 be 2 subspaces of Vector space.
$V(F) = \mathbb{R}^3(\mathbb{R})$

Now, $W_1 \cup W_2 = \{x : x = (a,0,0)$ or $x = (0,a,0)\ \forall\ a \in \mathbb{R}\}$

Now, We have to prove that $W_1 \cup W_2$ is not a subspace of $V(F)$
For this We have to prove that

$\alpha, \beta \in F$ & $x, y \in W_1 \cup W_2$
$\Rightarrow \alpha x + \beta y \notin W_1 \cup W_2$

let $x = (a,0,0)$ & $y = (0,b,0)$ where $a, b \in \mathbb{R}$
let $\alpha, \beta \in \mathbb{R}$

$$\begin{aligned}\alpha x + \beta y &= \alpha(a,0,0) + \beta(0,b,0) \\ &= (\alpha a, 0, 0) + (0, \beta b, 0) \\ &= (\alpha a, \beta b, 0) \\ &\notin W_1 \cup W_2\end{aligned}$$

Hence $W_1 \cup W_2$ is not a subspace of $V(F)$

<u>Theorem</u> The union of 2 subspace is a subspace iff one is Contained in the ~~oth~~ other.

<u>Proof</u>:

let W_1 & W_2 be two subspace of Vector Space $V(F)$ s.t.

$W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

then, We have to prove that $W_1 \cup W_2$ is a subspace of $V(F)$

$\because W_1 \subseteq W_2 \Rightarrow W_1 \cup W_2 = W_2$
$\therefore W_2$ is a subspace of $V(F)$
$\therefore W_1 \cup W_2$ is also a subspace of $V(F)$

Again, $W_2 \subset W_1 \Rightarrow W_1 \cup W_2 = W_1$
$\because W_1$ is a subspace of $V(F)$

$\therefore W_1 \cup W_2$ is also a subspace of $V(F)$

Hence, in either cases

$W_1 \cup W_2$ is a subspace of $V(F)$

Conversely suppose that $W_1 \cup W_2$ is a subspace of $V(F)$
then, we have to prove that $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

Let $W_1 \not\subset W_2$ and $W_2 \not\subset W_1$

$W_1 \not\subset W_2 \Rightarrow x \in W_1$ s.t. $x \notin W_2$

$W_2 \not\subset W_1 \Rightarrow y \in W_2$ s.t. $y \notin W_1$

Now,

$x \in W_1 \subset W_1 \cup W_2 \Rightarrow x \in W_1 \cup W_2$

& $y \in W_2 \subset W_1 \cup W_2 \Rightarrow y \in W_1 \cup W_2$

$\therefore x, y \in W_1 \cup W_2$

Since, $W_1 \cup W_2$ is a subspace of $V(F)$

$\therefore x, y \in W_1 \cup W_2 \Rightarrow x+y \in W_1 \cup W_2$

$\therefore x+y \in W_1$ or $x+y \in W_2$.

Since, W_1 is a subspace of $V(F)$

$\therefore x+y \in W_1,\ x \in W_1$

$\Rightarrow x+y \in W_1,\ -x \in W_1$

$= x+y-x \in W_1$

$= y \in W_1$

Which is contradiction.

Again,

$x+y \in W_2$ & $y \in W_2$

$= x+y \in W_2,\ -y \in W_2$ ($\because W_2$ is a subspace of $V(F)$)

$= x+y-y \in W_2$

$= x \in W_2$

Which is contradiction.