Date

Sequences

Def^n: A seq^n is an infinite progression of real nos.
It is denoted as $\langle a_n \rangle_{n=1}^{\infty}$ or $\langle a_n \rangle_{n \in \mathbb{N}}$ or simply $\langle a_n \rangle$

Bounded seq^{n}s

Def^n: $\langle a_n \rangle$ is s.t.b bounded above iff $\exists\ K \in \mathbb{R}$ s.t. $a_n \leq K\ \forall n \in \mathbb{N}$
In this case, K is called an upper bound of $\langle a_n \rangle$

Def^n: $\langle a_n \rangle$ is s.t.b bounded below iff $\exists\ k \in \mathbb{R}$ s.t $k \leq a_n\ \forall n \in \mathbb{N}$
In this case, k is called a lower bound of $\langle a_n \rangle$

Def^n: $\langle a_n \rangle$ is s.t.b bounded iff it is bounded above and bounded below
i.e $\exists\ K, k \in \mathbb{R}$ s.t. $k \leq a_n \leq K\ \ \forall\ n \in \mathbb{N}$

Note: $\langle a_n \rangle$ is bounded iff $\exists\ K > 0$ s.t.
$-K \leq a_n \leq K\ \ \forall\ n \in \mathbb{N}$
i.e $|a_n| \leq K\ \ \forall\ n \in \mathbb{N}$

Sup. and Inf. of seq^{n}s

Def^n: $\sup \langle a_n \rangle = u$ iff
(i) u is least upper bound of $\langle a_n \rangle$
(ii) $u - \epsilon$ is not an upper bound of $\langle a_n \rangle$
i.e $\exists\ n \in \mathbb{N}$ s.t. $a_n > u - \epsilon$.

Def^n: $\inf \langle a_n \rangle = v$ iff
(i) v is greatest lower bound of $\langle a_n \rangle$
(ii) $v + \epsilon$ is not an lower bound of $\langle a_n \rangle$
i.e $\exists\ n \in \mathbb{N}$ s.t. $a_n < v + \epsilon$

Date

Note: K is not an upper bound of $\langle a_n \rangle$ would mean that
$\exists$ $n \in \mathbb{N}$ s.t $a_n > K$

Monotonic seq^{n}s

Def^n: $\langle a_n \rangle$ is s.t.b monotonically increasing ($\uparrow$) iff
$a_1 \leq a_2 \leq a_3 \leq \cdots$
i.e $a_n \leq a_{n+1}$ $\forall$ $n \in \mathbb{N}$.

Eg: (i) $\langle 1, 3, 5 \ldots \rangle$
(ii) $\langle -1, -1/2, -1/3, \cdots \rangle$
(iii) $\langle 1, 2, 2, 3, 4, 4, \ldots \rangle$

Def^n: $\langle a_n \rangle$ is s.t.b monotonically decreasing ($\downarrow$) iff
$a_1 \geq a_2 \geq a_3 \geq \cdots$
i.e $a_n \geq a_{n+1}$ $\forall$ $n \in \mathbb{N}$.

Def^n: $\langle a_n \rangle$ is s.t.b monotonic iff $\langle a_n \rangle$ is either $\uparrow$ or $\downarrow$

Convergent seq^{n}s

Def^n: A seq^n $\langle a_n \rangle$ is said to converge to a number l, if given any $\epsilon > 0$, $\exists$ a +ve integer m (depending on ϵ) s.t.

$$|a_n - l| < \epsilon \quad \forall \; n \geq m.$$

i.e $l - \epsilon < a_n < l + \epsilon$, ie $a_n \in I(l, \epsilon)$

Eg: (i) $1/2, 2/3, 3/4 \cdots \rightarrow 1$
(ii) $0.9, 0.99, 0.999, \cdots \rightarrow 1$
(iii) $1.9, 1.99, 1.999 \cdots \rightarrow 2$
(iv) $1, 1, 1, 1 \cdots \rightarrow 1$

Page No. ____
Date ____

Thm: A seq^n cannot converge to more than one limit.

Proof:

let if possible, the seq^n $\langle a_n \rangle$ converges to two numbers say l and l'. $l \neq l'$, let $\epsilon > 0$ be any given no.

Since, $\langle a_n \rangle$ converges to l,

so $\exists$ +ve integer m_1 s.t.

$$|a_n - l| < \epsilon/2 \quad \forall \; n \geq m_1 \quad \text{—— (1)}$$

Since, $\langle a_n \rangle \to l'$, so, $\exists$ +ve integer m_2 s.t.

$$|a_n - l'| < \epsilon/2 \quad \forall \; n \geq m_2 \quad \text{—— (2)}$$

Now, $|l - l'| = |l - a_n + a_n - l'| \leq |l - a_n| + |a_n - l'|$

(add and subt. by a_n)

$$< \epsilon/2 + \epsilon/2 = \epsilon \quad \forall \; n \geq m$$

$$|l - l'| < \epsilon \quad \forall \; n \geq m$$

Since, ϵ is arbitrarily small,

$$|l - l'| = 0 \Rightarrow l = l'$$

Hence, seq^n can't converge to more than one limit.

Thm: Every convergent seq^n is bounded but converse is not true.

Proof:

let us suppose that $\langle a_n \rangle$ be convergent seq^n and

$$\lim_{n \to \infty} a_n = l$$

$\epsilon > 0$ be given, $\exists$ +ve integer m s.t.

$$|a_n - l| < \epsilon \quad \forall \; n \geq m$$

$$l - \epsilon < a_n < l + \epsilon \quad \forall \; n \geq m \quad \text{—— } (*)$$

let $K = \max.\{a_1, a_2, \ldots, a_{m-1}, l + \epsilon\}$ —— (1)

$k = \min.\{a_1, a_2, \ldots, a_{m-1}, l - \epsilon\}$ —— (2)

from (1), (2) and $(*)$

$$k \leq a_n \leq K \quad \forall \; n \Rightarrow \langle a_n \rangle \text{ is bounded.}$$

Page No. ____
Date ____

The converse of the thm need not be true.

for eg. $\langle (-1)^n \rangle = \langle -1, 1, -1, 1, \dots \rangle$ is bounded, but is not convergent

we have $\langle a_n \rangle = \langle -1, 1, -1, 1, \dots \rangle$

let if possible, $\lim_{n \to \infty} a_n = l$ (convergent to l)

then for $\epsilon > 0$, $\exists$ +ve integer m s.t.

$$|a_n - l| < \epsilon \quad \forall \; n \geq m$$

$$|1 - l| < \epsilon \quad \forall \; n \geq m \text{ \& } n \text{ is even}$$

$$|-1 - l| < \epsilon \quad \forall \; n \geq m \text{ and } n \text{ is odd}$$

Now, $2 = |-1-1|$

$$= |-1 + l + (-l - 1)|$$

$$\leq |-1 + l| + |-l - 1|$$

$$< 2\epsilon$$

let choose $\epsilon = 1/2$

$2 < 1$, which is contradiction.

Hence, $\langle (-1)^n \rangle$ is not convergent.

Monotone Convergence Thm

Thm: Every monotonic bounded seq^n converges.

Proof:

$\langle a_n \rangle$ is monotonic seq^n then it is either $\downarrow$ or $\uparrow$

Case(i) $\langle a_n \rangle$ is monotonically increasing.

As $\langle a_n \rangle$ is bounded above

$\therefore$ it has supremum, say u (By completness Axiom)

We assert that $\langle a_n \rangle \to u$.
i.e To show: for $\epsilon > 0$, $\exists$ m +ve integer s.t
$|a_n - u| < \epsilon \;\; \forall\; n \geq m$
i.e $u - \epsilon < a_n < u + \epsilon \;\; \forall\; n \geq m$

let $\epsilon > 0$ be given
Now u = least upper bound of $\langle a_n \rangle$
$\therefore$ $u - \epsilon$ is not an upper bound of $\langle a_n \rangle$
$\exists$ $m \in \mathbb{N}$ s.t $a_m > u - \epsilon$
As $\langle a_n \rangle$ is $\uparrow$
$\therefore$ $a_n \geq a_m \;\; \forall\; n \geq m$
$> u - \epsilon$
i.e $a_n > u - \epsilon \;\; \forall\; n \geq m$ — ①

Also, as u is an upper bound of $\langle a_n \rangle$
$\therefore$ $a_n \leq u \;\; \forall\; n \in \mathbb{N}$
$< \epsilon + u$
i.e $a_n < u + \epsilon \;\; \forall\; n \geq m$ — ②
① & ②
$\Rightarrow u - \epsilon < a_n < u + \epsilon \;\; \forall\; n \geq m$
$\therefore$ $\langle a_n \rangle \to u$.

Case ii) $\langle a_n \rangle$ is monotonically $\downarrow$ (similar)

Page No.
Date

Algebra of Convergence

Thm: If $\langle a_n \rangle \to a$ and $\langle b_n \rangle \to b$, then $\langle a_n + b_n \rangle \to a+b$

Proof:

Let $\epsilon > 0$ be given.

To show: $\exists\ m \in \mathbb{N}$ s.t. $|(a_n+b_n) - (a+b)| < \epsilon \quad \forall\ n \geq m$

Consider $|a_n + b_n - (a+b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|$ — (i)

Now as given

$\langle a_n \rangle \to a \quad \therefore\ \epsilon/2 > 0,\ \exists\ m_1 \in \mathbb{N}$ s.t.

$|a_n - a| < \epsilon/2 \quad \forall\ n \geq m_1$

& $\langle b_n \rangle \to b \quad \therefore\ \epsilon/2 > 0,\ \exists\ m_2 \in \mathbb{N}$ s.t.

$|b_n - b| < \epsilon/2 \quad \forall\ n \geq m_2$

Let $m = \max.\{m_1, m_2\}$

then clearly

$$\left.\begin{aligned} |a_n - a| &< \epsilon/2 \\ |b_n - b| &< \epsilon/2 \end{aligned}\right\} \quad \forall\ n \geq m$$

Hence by (i) $|a_n + b_n - (a+b)| < \epsilon \quad \forall\ n \geq m$

$\therefore\ \langle a_n + b_n \rangle \to a+b$

Thm If $\langle a_n \rangle \to b$ and $\langle b_n \rangle \to b$, then

$\langle a_n - b_n \rangle \to a-b$

Proof: Same as above

by using $|x-y| \leq |x| + |y|$

Thm : Scalar multiple thm

If $\langle a_n \rangle \to a$, then $\langle k a_n \rangle \to ka$, $k \neq 0$ constant.
only statement.

Thm : (Product thm)
If $\langle a_n \rangle \to a$ & $\langle b_n \rangle \to b$, then $\langle a_n b_n \rangle \to ab$.

Proof :
Given $\langle a_n \rangle \to a$ & $\langle b_n \rangle \to b$
$\therefore \exists\ \epsilon_1 > 0$ and $\epsilon_2 > 0$ s.t. +ve integer m_1, m_2 s.t.
$|a_n - a| < \epsilon_1 \ \forall\ n \geq m_1$
$|b_n - b| < \epsilon_2 \ \forall\ n \geq m_2$

let $m = \max.\{m_1, m_2\}$
$$\left.\begin{aligned} |a_n - a| &< \epsilon_1 \\ |b_n - b| &< \epsilon_2 \end{aligned}\right\} \ \forall\ n \geq m \quad \text{— } ⊛$$

To show: $\langle a_n b_n \rangle \to ab$
i.e $|a_n b_n - ab| < \epsilon \ \forall\ n \geq m$

Consider. $|a_n b_n - ab| = |a_n b_n - ab + a_n b - a_n b|$
$= |a_n (b_n - b) + b(a_n - a)|$
$\leq |a_n||b_n - b| + |b| \cdot |a_n - a|$

Now: Seq^n $\langle a_n \rangle$ being convergent is bounded.
i.e $\exists$ a +ve integer K s.t.
$|a_n| \leq K \ \forall\ n$ — (3)
by (1), (2) & (3) A becomes
$|a_n b_n - ab| < K\epsilon_2 + |b|\epsilon_1$
choose $\epsilon_2 = \epsilon/2K$ and $\epsilon_1 = \dfrac{\epsilon}{2(|b|+1)}$

we get $|a_nb_n - ab| < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon \quad \forall n \geq m$

Hence, $\langle a_n b_n \rangle \to ab$.

Thm: Modulus Thm $(a \neq 0)$

If $\langle a_n \rangle \to a$, then $\langle |a_n| \rangle \to |a|$

Proof: Only statement.

* Converse of above thm is not true.

eg: $\langle a_n \rangle \Rightarrow \langle 1, -1, 1, -1, \ldots \rangle$

$\langle |a_n| \rangle = \langle 1, 1, \ldots \rangle$

then $\langle |a_n| \rangle$ converges, but $\langle a_n \rangle$ does not converges.

Thm: If $\langle a_n \rangle \to 0$ and $\langle b_n \rangle$ is bounded then

$\langle a_n b_n \rangle \to 0$

(Only statement)

Thm: let $\langle a_n \rangle \to l$, if $a_n \geq 0 \ \forall n$, then $l \geq 0$

Proof:

If Possible, suppose $l < 0$

clearly, we can choose $\epsilon > 0$ so small that $l + \epsilon < 0$ — (1)

As $\langle a_n \rangle \to l \ \therefore \ \epsilon > 0, \exists \ m \in \mathbb{N}$ s.t.

$l - \epsilon < a_n < l + \epsilon < 0 \quad \forall n \geq m$

Thus $a_n < 0 \quad \forall n \geq m$

contradiction

$\therefore l \geq 0$.

Order Preservation thm.

Thm: Let $\langle a_n \rangle \to a$ and $\langle b_n \rangle \to b$

If $a_n \leq b_n \ \forall n \in \mathbb{N}$ then $a \leq b$

Proof:

$$a_n \leq b_n \quad \forall n \in \mathbb{N}$$

$$\Rightarrow b_n - a_n \geq 0 \quad \forall n \in \mathbb{N}$$

Also. $\langle b_n - a_n \rangle \to b - a$ (By difference Thm)

$\therefore$ By last thm

$$b - a \geq 0$$

i.e $a \leq b$.

Thm: Sandwich thm

let $a_n \leq b_n \leq c_n \quad \forall n \in \mathbb{N}$

If $\langle a_n \rangle \to l$ and $\langle c_n \rangle \to l$, then $\langle b_n \rangle \to l$

Proof:

Let $\epsilon > 0$ be given

to show $\exists\, m \in \mathbb{N}$ s.t. $l - \epsilon < b_n < l + \epsilon \quad \forall n \geq m$

As $\langle a_n \rangle \to l$ $\therefore$ corr. to $\epsilon > 0$, $\exists\, m_1 \in \mathbb{N}$ s.t.

$$l - \epsilon < a_n < l + \epsilon \quad \forall n \geq m_1$$

As $\langle c_n \rangle \to l$ $\therefore$ $\epsilon > 0$, $\exists\, m_2 \in \mathbb{N}$ s.t.

$$l - \epsilon < c_n < l + \epsilon \quad \forall n \geq m_2$$

let $m = \max.\{m_1, m_2\}$

then $\left.\begin{array}{l} l - \epsilon < a_n < l + \epsilon \\ l - \epsilon < c_n < l + \epsilon \end{array}\right\} \quad \forall n \geq m \quad - (1)$

Also given $a_n \leq b_n \leq c_n \quad - (2)$

(1) & (2) $\Rightarrow l - \epsilon \leq b_n < l + \epsilon \quad \forall n \geq m$

$\therefore \langle b_n \rangle \to l$.

Page No.
Date

DIVERGENCE & OSCILLATION

Def^n: (i) $\langle a_n\rangle \to \infty$ iff for each $K>0$, $\exists\, m \in \mathbb{N}$ s.t.
$a_n > K \quad \forall\, n \geq m$

(ii) $\langle a_n\rangle \to -\infty$ iff for each $K>0$, $\exists\, m \in \mathbb{N}$ s.t.
$a_n < -K \quad \forall\, n \geq m$

(iii) $\langle a_n\rangle$ is s.t.b divergent iff $\langle a_n\rangle \to \infty$ or $\langle a_n\rangle \to -\infty$

Terminology $\langle a_n\rangle \to \infty$ may also be written as $\lim\limits_{n\to\infty} a_n = \infty$

$\langle a_n\rangle \to -\infty$ may also be written as $\lim\limits_{n\to\infty} a_n \to -\infty$

Eg of divergent seq^n
(i) $1, 3, 5, 7, \dots \to \infty$
(ii) $-1, -3, -5, -7, \dots \to -\infty$

Note: Every divergent seq^n is unbounded

Def^n: $\langle a_n\rangle$ is s.t.b oscillatory iff $\langle a_n\rangle$ neither converges nor diverges.

Eg: (i) $1, 2, 1, 2 \dots$
(ii) $-1, 1, -2, 2, \dots$
(iii) $2, 1/2, 3, 1/3, 4, 1/4 \dots$

Note (i) A convergent seq^n has a finite limit
(ii) " divergent " " an infinite limit. $(\pm\infty)$
(iii) " oscillatory " " no limit

Page No.
Date

* Eg to show that if $\langle a_n \rangle \to \infty$ and $\langle b_n \rangle \to -\infty$ then $\langle a_n + b_n \rangle$ may
(i) diverge to ∞ (ii) diverge to $-\infty$ (iii) converges
(iv) oscillate.

① $\langle a_n \rangle = 2, 4, 6 \ldots \quad \to \infty$
$\langle b_n \rangle = -1, -2, -3 \ldots \quad \to -\infty$
$\langle a_n + b_n \rangle = 1, 2, \ldots \quad \to \infty$ (diverge)
$\langle b_n - a_n \rangle = -3, -5, -9 \quad \to -\infty$

② $\langle a_n \rangle = 1, 2, 3 \ldots \quad \to \infty$
$\langle b_n \rangle = -1, -2, -3 \quad \to -\infty$
$\langle a_n + b_n \rangle = 0, 0, 0, \ldots \quad \to 0$ (Converges)

③ $\langle a_n \rangle = 2, 4, 6, 8, \ldots \quad \to \infty$
$\langle b_n \rangle = -1, -2, -5, -6 \quad \to -\infty$
$\langle a_n + b_n \rangle = 1, 2, 1, 2 \ldots$ oscillates.

Problems based on def^n of convergence

Ques 1.) Show that $\langle 1/n^2 \rangle \to 0$

Sol^n:

let $\epsilon > 0$ be given

Consider $|1/n^2 - 0| = 1/n^2 < \epsilon$ if $n^2 \epsilon > 1$
i.e $n > 1/\sqrt{\epsilon}$

let m be a +ve integer s.t. $m > 1/\sqrt{\epsilon}$
then
$|1/n^2 - 0| < \epsilon \quad \forall \; n \geq m$
$\therefore \langle 1/n^2 \rangle \to 0$.

DELTA Notebook

Date

Ques 2. Show that $\langle n/n+1 \rangle \to 1$.

Solⁿ Let $\epsilon > 0$ be given

$$\left|\frac{n}{n+1} - 1\right| = \frac{1}{n+1} < \epsilon \text{ if } (n+1)\epsilon > 1$$

$$\text{i.e } n > \frac{1}{\epsilon} - 1$$

let m be positive integer s.t. $m > \frac{1}{\epsilon} - 1$

then $\left|\frac{n}{n+1} - 1\right| < \epsilon \quad \forall\, n \geq m$

$\therefore \left\langle \frac{n}{n+1} \right\rangle \to 1$

Ques 3. Show that the seqⁿ $\langle a_n \rangle$ where

$$a_n = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(2n)^2}$$

Converges to zero,

Solⁿ: Let $\epsilon > 0$ be given

$$|a_n - 0| = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(2n)^2}$$

$$< \underbrace{\frac{1}{n^2} + 1/n^2 + \cdots + 1/n^2}_{n \text{ terms}}$$

$$= n/n^2 = \frac{1}{n} < \epsilon \text{ if } n > 1/\epsilon$$

let m be a +ve integer st $m > 1/\epsilon$

then $|a_n - 0| < \epsilon \quad \forall\, n \geq m$

$\therefore \langle a_n \rangle \to 0$

Date

Ques: If $|a|<1$, show $\lim_{n\to\infty} a^n = 0$

Sol^n

Let $\epsilon>0$ be given

Consider $|a^n - 0| = |a^n| < \epsilon \Rightarrow |a|^n < \epsilon$

iff. $n \log|a| < \log \epsilon$

i.e $n \geq \frac{\log \epsilon}{\log|a|}$ $\left(\because |a|<1 \Rightarrow \log|a|<0\right)$

let m be a +ve integer s.t.

$m > \frac{\log \epsilon}{\log|a|}$

then $|a^n - 0| < \epsilon \ \forall\ n \geq m$

$\therefore \langle a^n \rangle \to 0$

Ques: Show that $\lim_{n\to\infty} n^{1/n} = 1$ or $\langle n^{1/n} \rangle \to 1$

Sol^n $n^{1/n} \geq 1 \quad \forall\ n \geq N$

(Reason If possible suppose $n^{1/n} < 1$

$\therefore n < 1^n$ i.e $n<1$ (false))

Define $\langle b_n \rangle$ as : $b_n = n^{1/n} - 1$ then $b_n \geq 0 \ \forall\ n \in \mathbb{N}$

To show: $\lim_{n\to\infty} b_n = 0$

let $\epsilon>0$ be given

To show: $\exists\ m \in \mathbb{N}$ s.t. $|b_n - 0| < \epsilon \ \forall\ n \geq m$

Now $n^{1/n} = 1 + b_n$

$n = (1+b_n)^n$

$= 1 + {}^nC_1 b_n + {}^nC_2 b_n^2 + \cdots + b_n^n$

$$\geq 1 + {}^nC_2\, b_n^2$$

$$\Rightarrow n-1 \geq \frac{n(n-1)}{2!} b_n^2$$

$$\Rightarrow b_n \leq \sqrt{\frac{2}{n}} < \epsilon \text{ if } \frac{2}{n} < \epsilon^2 \text{ ie } n > 2/\epsilon^2$$

let m be a +ve integer s.t. $m > 2/\epsilon^2$

then clearly $b_n < \epsilon \ \forall\ n \geq m$

$\therefore \lim\limits_{n\to\infty} b_n = 0$

i.e $\lim\limits_{n\to\infty} (n^{1/n} - 1) = 0$

i.e $\lim\limits_{n\to\infty} n^{1/n} = 1$

Problems Based on Monotonic Congt Thm

Ques: show that seq^n $\langle a_n \rangle$ where $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$ is convergent.

Sol^n:

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

$$a_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n+2}$$

$$a_{n+1} - a_n = \frac{1}{2n+1} + \frac{1}{2n+1} - \frac{1}{n+1}$$

$$\Rightarrow \frac{1}{2n+1} - \frac{1}{2n+2} > 0 \quad \forall\ n \in \mathbb{N}$$

Page No. ___
Date ___

i.e $a_n > a_{n+1}$ $\forall n \in \mathbb{N}$

$\therefore \langle a_n \rangle$ is $\uparrow$ — ①

Bdd above.

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

$$< \frac{1}{n} + 1/n + \cdots + 1/n = n \cdot 1/n = 1$$

i.e $a_n < 1$ $\forall n \in \mathbb{N}$

$\langle a_n \rangle$ is bdd above — ②

from ① & ②

$\Rightarrow \langle a_n \rangle$ is convergent.

Ques: Show that the seq^n $\langle a_n \rangle$ where

$a_1 = 1$, $a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{(n-1)!}$ $(n \geq 2)$ converges.

Sol^n.

$$a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{(n-1)!}$$

$$a_{n+1} = 1 + 1/1! + 1/2! + \cdots + \frac{1}{n!}$$

$$a_{n+1} - a_n = 1/n! > 0 \quad \forall n \in \mathbb{N}$$

$\therefore a_n < a_{n+1}$

$\therefore \langle a_n \rangle$ is $\uparrow$ — ①

Bdd Above: $a_n = 1 + 1/1! + 1/2! + \cdots + 1/n!$

$= 1 + 1 + 1/2 + 1/2 \cdot 3 + \cdots$ n terms

$\leq 1 + 1 + 1/2 + 1/2 \cdot 1/2 + 1/2 \cdot 1/2 \cdot 1/2 + \cdots$ n term

Page No.
Date

$= 1 + \{1 + 1/2 + 1/2^2 + \cdots \quad n-1 \text{ terms})$

$= 1 + \dfrac{1(1-(1/2)^{n-1})}{1-1/2}$

$= 1 + 2\left[1 - \left(\frac{1}{2}\right)^{n-1}\right] = 3 - \dfrac{2}{2^{n-1}} < 3$

Thus,

$a_n < 3 \quad \forall\, n \in \mathbb{N}$

Thus $\langle a_n \rangle$ is bdd above — (2)

from (1) & (2) $\Rightarrow$ $\langle a_n \rangle$ is convergent.

Problems On Applications of sandwich Thm

Ques: show $\lim\limits_{n\to\infty}\left[\dfrac{1}{\sqrt{n^2+1}} + \dfrac{1}{\sqrt{n^2+2}} + \cdots + \dfrac{1}{\sqrt{n^2+n}}\right] = 1$

Sol^n:

Let $a_n = \dfrac{1}{\sqrt{n^2+1}} + \dfrac{1}{\sqrt{n^2+2}} + \cdots + \dfrac{1}{\sqrt{n^2+n}}$

$< \dfrac{1}{\sqrt{n^2}} + \dfrac{1}{\sqrt{n^2}} + \cdots + \dfrac{1}{\sqrt{n^2}} = \dfrac{n}{\sqrt{n^2}} = 1$

Also,

$a_n > \dfrac{1}{\sqrt{n^2+n}} + \dfrac{1}{\sqrt{n^2+n}} + \cdots + \dfrac{1}{\sqrt{n^2+n}} = \dfrac{n}{\sqrt{n^2+n}} = \dfrac{1}{\sqrt{1+1/n}}$

Thus $\dfrac{1}{\sqrt{1+1/n}} \le a_n \le 1 \quad \forall\, n \in \mathbb{N}$

As $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{1+1/n}} = 1$ & $\lim\limits_{n\to\infty} 1 = 1$

Date

$\therefore \lim_{n\to\infty} a_n = 1$ (By Sandwich thm)

Ques: Show $\lim_{n\to\infty}\left[\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\cdots+\frac{1}{(2n)^2}\right]=0$

Solⁿ:

Let $a_n = \frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\cdots+\frac{1}{(2n)^2}$

$\leq \frac{1}{n^2}+\frac{1}{n^2}+\cdots+\frac{1}{n^2} = \frac{n}{n^2} = \frac{1}{n}$

Thus $0 \leq a_n \leq \frac{1}{n} \quad \forall\, n \in \mathbb{N}$

As $\lim_{n\to\infty} 0 = 0$, $\lim_{n\to\infty}\frac{1}{n} = 0$ $\therefore \lim_{n\to\infty} a_n = 0$

Ques: Show $\lim_{n\to\infty}\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}+\cdots+\frac{1}{(2n)^2}\right]=0$

do yourself.

Ques: Show that the seqⁿ $\langle a_n\rangle$ defined as $a_n = \frac{1}{(2n+1)^2}+\frac{1}{(2n+2)^2}+\cdots+\frac{1}{(3n)^2}$ is convergent o.

Solⁿ:

$a_n = \frac{1}{(2n+1)^2}+\frac{1}{(2n+2)^2}+\cdots+\frac{1}{(3n)^2}$

$< \frac{1}{(2n)^2}+\frac{1}{(2n)^2}+\cdots+\frac{1}{(2n)^2} = \frac{n}{4n^2} = \frac{1}{4n}$

Thus $0 \leq a_n \leq \frac{1}{4n}$

As $\lim_{n\to\infty} 0 = 0$ & $\lim_{n\to\infty} \frac{1}{4n} = 0$ $\therefore \lim_{n\to\infty} a_n = 0$

$\therefore \langle a_n\rangle$ is convergent (to zero)