Numerical Integration $\Rightarrow$ general problem of numerical integration is to find an approximate value of the ~~integration~~ integral

$$I = \int_a^b w(x) f(x)\, dx \quad \text{— (1)}$$

where $w(x) > 0$ in $(a\ b)$ is the weight function. we assume that $w(x)$ and $w(x) f(x)$ are integrable in the Riemann sense on $[a\ b]$, limit of integration may be finite, semi-finite or infinite. The integral (1) is approximated by a finite linear combination of value of $f(x)$ in the form-

$$I = \int_a^b w(x) f(x)\,dx \simeq \sum_{k=0}^{n} \lambda_k f(x_k) \quad -(2)$$

where x_k, $k = 0$ to n are called the abscissas or nodes distributed within the limits of integration $[a, b]$ and λ_k, $k = 0$ to n are called weights of the integration rule or the quadrature formula (2). The error of approximation is given as

$$R_n = \int_a^b w(x) f(x)\,dx - \sum_{k=0}^{n} \lambda_k f(x_k) \quad -(C)$$

Definition :- An Integration method of the form (2) is said to be of order p if it produces exact result ($R_n = 0$) for all polynomials of degree less than or equal to p

in (2) we have $2n+2$ unknowns ($n+1$ nodes x_k's and $n+1$ weights λ_k's) and the method can be made exact for polynomials of degree $\leq 2n+1$. Thus, the method of the form (2) can be of maximum order $2n+1$. If some of the nodes are known in advance then order will be reduced. If all the $n+1$ abscissas are known in advance then we have to determine only $n+1$ weights, and the corresponding method will be of maximum order n.

Method. Based on Interpolation :- Given $n+1$ abscissas, x_k's and the corresponding values $f(x_k)$ the Lagrange Interpolating Polynomial fitting the data $(x_k, f(x_k))$, $k = 0$ to n is given by

$$f(x) = \sum_{k=0}^{n} l_k(x) f(x_k) + \frac{\pi(x)}{\lfloor n+1} f^{(n+1)}(\xi_e)$$

$$x_0 < \xi_e < x_n$$

where $l_k(x)$ is the lagrange fundamental polynomial — ③

$$l_k(x) = \frac{\pi(x)}{(x-x_k)\,\pi'(x_k)}$$ and $\pi(x) = (x-x_0)(x-x_1) \cdots (x-x_n)$

we replace the function $f(x)$ in ① by the interpolating polynomial ③ and Integrate within the given limits, we obtain

$$I = \int_a^b w(x) f(x)\, dx = \sum_{k=0}^{n} \left[\int_a^b w(x) l_k(x)\, dx\right] f(x_k) + \int_a^b w(x) \frac{\pi(x)}{\lfloor n+1} f^{(n+1)}(\xi_e)\, dx$$

$$= \sum_{k=0}^{n} \lambda_k f(x_k) + R_n \quad \text{— ④}$$

where $\lambda_k = \int_a^b w(x) l_k(x)\, dx$ — ⑤

and $R_n = \frac{1}{\lfloor n+1} \int_a^b w(x)\, \pi(x)\, f^{(n+1)}(\xi_e)\, dx$ — ⑥

Determination of the error term $\Rightarrow$ If $\pi(x)$ does not change sign in $[a\, b]$ and $f^{(n+1)}(x)$ is continuous in $[a\, b]$, then using the mean Value Theorem of Integral Calculus, we can write the error of approximation ⑥ in the form

$$R_n = \frac{f^{(n+1)}(\eta)}{\lfloor n+1} \int_a^b w(x)\, \pi(x)\, dx, \quad \eta \in (a\, b) \quad \text{— ⑦}$$

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If $\pi(x)$ changes sign in $[a, b]$, then we can obtain $|R_n|$ from (6) as

$$|R_n| \le \frac{1}{\lfloor n+1} \int_a^b w(x)\, |\pi(x)|\, |f^{(n+1)}(\xi_x)|\, dx$$

$$\le \frac{M_{n+1}}{\lfloor n+1} \int_a^b w(x)\, |\pi(x)|\, dx$$

where $|f^{(n+1)}(x)| \le M_{n+1}$, $x \in [a, b]$

Alternate method $\Rightarrow$ The error term can be also be obtained in the following manner.

$\because$ the method is exact for polynomials of degree $\le n$, we have

$R_n = 0$ when $f(x) = x^i$, $i = 0, 1, \ldots n$

and $R_n \ne 0$ when $f(x) = x^{n+1}$

Thus we can write the error term in the form

$$R_n = \frac{C}{\lfloor n+1} f^{(n+1)}(h) \quad \text{—(A)}$$

where $C = \int_a^b w(x)\, x^{n+1}\, dx - \sum_{k=0}^{n} \lambda_k x_k^{n+1}$ —(B)

is called the error constant, if $C = 0$ for x^{n+1} then we take the next term x^{n+2}

neglecting the error in (4) we get the integration method

$$\int_a^b w(x) f(x)\, dx = \sum_{k=0}^{n} \lambda_k f(x_k) \quad \text{—(8)}$$

Newton-Cotes methods: $\Rightarrow$ when $w(x) = 1$ and nodes x_k's are equispaced with $x_0 = a$, and $x_n = b$ with spacing $h = \frac{(b-a)}{n}$ the method (8) are called Called Newton-Cotes Integration method, The weights λ_k's are called Cotes number.

Put $x = x_0 + sh$, we get

$$\pi(x) = h^{n+1} s(s-1) \cdots (s-n)$$

$$l_k(x) = \frac{(-1)^{n-k}}{\lfloor k \lfloor n-k} s(s-1) \cdots (s-k+1)(s-k-1) \cdots (s-n)$$

$$\boxed{\begin{aligned} \lambda_k &= \frac{(-1)^{n-k}}{\lfloor k \lfloor n-k} h \int_0^n s(s-1) \cdots (s-k+1)(s-k-1) \cdots (s-n)\, ds \\ R_n &= \frac{h^{(n+2)}}{\lfloor n+1} \int_0^n s(s-1) \cdots (s-n) f^{(n+1)}(\xi)\, ds \quad \text{--- (9)} \end{aligned}}$$

--- (10)

For $n = 1$, we have $x_0 = a$, $x_1 = b$, $h = b-a$

$$\lambda_0 = -h \int_0^1 (s-1)\, ds = \frac{h}{2}$$

$$\lambda_1 = h \int_0^1 s\, ds = \frac{h}{2}.$$

and we get

$$\int_a^b f(x)dx = \lambda_0 f(x_0) + \lambda_1 f(x_1)$$

$$= \frac{h}{2} f(a) + \frac{h}{2} f(b)$$

$$= \frac{h}{2}[f(a) + f(b)]$$

$$\int_a^b f(x)dx = \frac{(b-a)}{2}[f(a) + f(b)] \quad \text{--- (11)}$$

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which is called the trapezoidal rule

The error in the trapezoidal rule becomes

$$R_1 = \frac{h^3}{2}\int_0^1 s(s-1) f''(\xi_x)\, ds$$

$\therefore$ Since $s(s-1)$ does not sign in $[0\ 1]$, we get

$$\boxed{R_1 = \frac{h^3}{2} f''(\eta)\int_0^1 s(s-1)\, ds = -\frac{h^3}{12} f''(\eta) = -\frac{(b-a)^3}{12} f''(\eta) \quad \eta \in (0,1)}$$

Thus the trapezoidal rule is exact for polynomial of degree ≤ 1 and is of order 1.

Alternatively $\Rightarrow$ we get from (B) we get

$$c = \int_a^b x^2\, dx - \frac{h}{2}\left[\lambda_0 x_0^2 + \lambda_1 x_1^2\right]$$

$$= \int_a^b x^2 dx - \frac{1}{2}\left((b-a)\right)(b^2+a^2)$$

$$= \frac{1}{3}(b^3-a^3) - \frac{1}{2}(b-a)(b^2+a^2)$$

$$c = -\frac{1}{6}(b-a)^3$$

$\therefore$ from (A)

$$\boxed{R_1 = -\frac{(b-a)^3}{12} f''(\eta) = -\frac{h^3}{12} f''(\eta)}$$

for $n=2$, we have $h = \frac{(b-a)}{2}$, $x_0 = a$,

$x_1 = \frac{a+b}{2}$

from (9) we get $\quad x_2 = b$

$$\lambda_0 = \frac{h}{2}\int_0^2 (s-1)(s-2)\, ds = \frac{h}{3}$$

$$\lambda_1 = -h\int_0^2 s(s-2)\,ds = \frac{4h}{3}$$

$$\lambda_2 = \frac{h}{2}\int_0^2 s(s-1)\,ds = \frac{h}{3}$$

and we get

$$\boxed{I = \int_a^b f(x)\,dx = \left(\frac{b-a}{6}\right)\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]} \quad -(13)$$

which is called the Simpson's rule

the error associated with this method is given by

$$R_2 = \frac{h^4}{\lfloor 3}\int_0^2 s(s-1)(s-2)\,f'''(\xi)\,ds$$

Note that $s(s-1)(s-2)$ changes sign in $(0,2)$ we use the equation (B) to obtain the error term, since the method is exact for polynomials of degree ≤ 2 we have.

$$C = \int_a^b x^3\,dx - \left(\frac{b-a}{6}\right)\left[a^3+4\left(\frac{a+b}{2}\right)^3+b^3\right]$$

$$= \frac{1}{4}(b^4-a^4) - \left(\frac{b-a}{12}\right)\left[2a^3+(a+b)^3+2b^3\right]$$

$$C = \frac{1}{4}(b^4-a^4) - \frac{1}{4}(b^4-a^4) = 0$$

This shows that the method (13) is exact for polynomials of degree three also. Hence the error term becomes

$$R_2 = \frac{C}{\lfloor 4} f^{(iv)}(\eta),\ \eta \in (0,2)$$

$$R_2 = \frac{C}{\lfloor 4} f^{(iv)}(\eta),\ \eta \in (0,2)$$

where

$$C = \int_a^b x^4 dx - \frac{(b-a)}{6}\left[a^4 + 4\left(\frac{a+b}{2}\right)^4 + b^4\right]$$

$$= \frac{1}{5}(b^5 - a^5) - \left(\frac{b-a}{24}\right)\left[4a^4 + (a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4) + 4b^4\right]$$

$$C = -\frac{(b-a)^5}{120}$$

Therefore, the error of approximation in the Simpson's rule become

$$R_2 = -\frac{(b-a)^5}{2880} f^{(IV)}(h) = -\frac{h^5}{90} f^{IV}(h)$$

when $n=3$, the corresponding integration method is called 3/8th Simpson's rule, the weights λ_k of the integration method (B) with $w(x)=1$ for $n \le 6$ are given

$n \backslash \lambda$	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
1	1/2	1/2					
2	1/3	4/3	1/3				
3	3/8	9/8	9/8	3/8			
4	14/45	64/45	24/45	64/45	14/45		
5	95/288	375/288	250/288	250/288	375/288	95/288	
6	41/140	216/140	27/140	272/140	27/140	216/140	41/140

The methods of the form (B) include the end points x_0 and x_n as abscissas, such method are also called closed-type methods.
The methods which do not include the end points as abscissas are often called open-type methods.

Open Type Integration Rules $\Rightarrow$ Replace $f(x)$ in (1) by Lagrange interpolating polynomial fitting the $n-1$ data points $(x_k, f(x_k))$, $k = 1$ to $(n-1)$ and integrate between the given limits. Some of the open-type integration methods $(w(x) = 1)$ together with the associated errors are listed below, the nodes are equispaced with $h = \frac{b-a}{n}$ and $x_0 = a$, $x_n = b$

(I) Mid-Point rule $(n=2)$, $x_0 = a$, x_0+h, $x_0+2h = b$

$$\int_a^b f(x)dx = 2h f(x_0+h) + \frac{h^3}{3} f''(\xi_1)$$

(II) Two-point rule $(n=3)$, $x_0 = a$, x_0+h, x_0+2h, $x_0+3h = b$

$$\int_a^b f(x)dx = \frac{3h}{2}\left[f(x_0+h) + f(x_0+2h)\right] + \frac{3}{4} h^3 f''(\xi_2)$$

(III) Three-point rule $(n=4)$, $x_0 = a$, x_0+h, x_0+2h, x_0+3h, $x_0+4h = b$

$$\int_a^b f(x)dx = \frac{4h}{3}\left[2f(x_0+h) - f(x_0+2h) + 2f(x_0+3h)\right] + \frac{14h^5}{45} f^{iv}(\xi_3)$$

where $a < \xi_1, \xi_2, \xi_3 < b$

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Q-7 find the approximate value of

$$I = \int_0^1 \frac{dx}{1+x}$$

using (i) trapezoidal rule (ii) Simpson's rule obtain a bound for the errors, the exact value of $I = \log 2 = 0.693147$ correct to six decimal places

Solution ⇒ Using trapezoidal rule, we have

$$I \simeq \frac{1}{2}\left(1+\frac{1}{2}\right) = 0.75$$

$$\text{Error} = 0.75 - 0.693147 = 0.056853$$

The error in the trapezoidal rule is given by

$$|R_1| \le \frac{(b-a)^3}{12} \max_{0\le x\le 1} |f''(x)| \le \frac{1}{12} \max_{0\le x\le 1} \left|\frac{2}{(1+x)^3}\right|$$

$$\le \frac{1}{6}$$

Using the Simpson's rule, we have

$$I \simeq \frac{1}{6}\left(1+\frac{8}{3}+\frac{1}{2}\right) = \frac{25}{36} = 0.694444$$

$$\text{Error} = 0.694444 - 0.693147 = 0.001297$$

The error in the Simpson's rule is given by

$$|R_2| \le \frac{(b-a)^5}{2880} \max_{0\le x\le 1} |f^{iv}(x)| \le \frac{1}{2880} \max_{0\le x\le 1} \left|\frac{24}{(1+x)^5}\right|$$

$$= 0.008333$$

the actual error is much smaller than the error bound obtained from theoretical considerations

Q-2 Find the approximate value of $I = \int_0^1 \frac{\sin x}{x} dx$

① using ① mid-point rule ⑪ Two-Point rule

① Mid-point Rule ⇒ we have $h = \frac{b-a}{2} = \frac{1}{2}$

Therefore $x_0 = 0,\ x_1 = \frac{1}{2},\ x_2 = 1$

$I = \int_0^1 f(x)\,dx = 2hf(x_0+h) = f(\frac{1}{2}) = 2\sin(\frac{1}{2})$

⑪ Two-Point rule ⇒ we have $= 0.9589$

$x_0 = 0,\ x_1 = \frac{1}{3},\ x_2 = \frac{2}{3}$ and $x_3 = 1$ $\quad h = (\frac{b-a}{3}) = \frac{1}{3}$

$I = \int_0^1 f(x)\,dx = \frac{3h}{2}[f(x_0+h) + f(x_0+2h)]$

$= \frac{1}{2}[f(\frac{1}{3}) + f(\frac{2}{3})] = \frac{1}{2}[3\sin(\frac{1}{3}) + \frac{3}{2}\sin(\frac{2}{3})]$

$I = 0.9546$

Q-3 Find the remainder of the Simpson three eight rule

$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)]$

for equally spaced points $x_i = x_0 + ih$, $i = 1, 2, 3$. Use this Rule to approximate the value of the integral $I = \int_0^1 \frac{dx}{1+x}$. Also find a bound on the error using © and definition of the order, we can show that the rule is exact for $f(x) = 1, x, x^2, x^3$.

Now for $f(x) = 1$

$\int_{x_0}^{x_3} dx = (x_3 - x_0) = 3h = \frac{3h(8)}{(8)} = 3h.$

which is true, the error Constant is given by

$$C = \int_{x_0}^{x_4} x^4 dx - \frac{3h}{8}[x_0^4 + 3x_1^4 + 3x_2^4 + x_3^4]$$

$$= \frac{1}{5}[(x_0+3h)^5 - x_0^5] - \frac{3h}{8}[x_0^4 + 3(x_0+h)^4 + 3(x_0+2h)^4 + (x_0+3h)^4]$$

$$= \frac{1}{5}[x_0^5 + 15x_0^4 h + 90x_0^3 h^2 + 270x_0^2 h^3 + 405x_0 h^4 + 243h^5 - x_0^5] - \frac{3h}{8}[x_0^4 + 3[x_0^4 + 4x_0^3 h + 4x_0^3 h + 6x_0^2 h^2 + 4x_0 h^3 + h^4] + 3(x_0^4 + 8hx_0^3 + 24x_0^2 h^2 + 32x_0 h^3 + 16h^4) + (x_0^4 + 12x_0^3 h + 54x_0^2 h^2 + 108x_0 h^3 + 81h^4)$$

$$= \left[\frac{243}{5} - \frac{99}{2}\right] h^5 = -\frac{9}{10} h^5$$

Therefore the error term is given by

$$E = \frac{C}{4!} f^{iv}(h) = -\frac{9h^5}{10\times 24} f^{iv}(h)$$

$$= -\frac{3}{80} h^5 f^{iv}(h) \quad , x_0 < h < x_3$$

we have $x_0 = 0$, $x_1 = 1/3$, $x_2 = 2/3$, $x_3 = 1$

$\Rightarrow h = 1/3$

$$I = \frac{3}{8}\left(\frac{1}{3}\right)\left(f(0) + 3f\left(\frac{1}{3}\right) + 3f\left(\frac{2}{3}\right) + f(1)\right)$$

$= \frac{1}{8}\left[1 + \frac{9}{4} + \frac{9}{5} + \frac{1}{2}\right] = 0.69375$

Gauss Quadrature method $\Rightarrow$ we know that in the integration

$$\int_a^b w(x) f(x) dx = \sum_{k=0}^{n} \lambda_k f(x_k), \quad (1)$$

when the nodes are also known then corresponding methods are called Newton-Cotes method, if nodes are also to be determined, then the method are called Gaussian Integration methods.

Any using the transformation $x = \left(\frac{b-a}{2}\right)t + \frac{(b+a)}{2}$ we can transformed the finite interval $[a, b]$ in the $[-1, 1]$, and the equation (1) can be transformed

$$\int_{-1}^{1} w(x) f(x) dx = \sum_{k=0}^{n} \lambda_k f(x_k) \quad (11)$$

where $w(x) > 0$, $-1 \le x \le 1$ is the weight function

Gauss-Legendre Integration method $\Rightarrow$

~~Substitute~~ Put $w(x) = 1$ in the equation (11) we get

$$\int_{-1}^{1} f(x) dx = \sum_{k=0}^{n} \lambda_k (f(x_k)) \quad (111)$$

in this case all the nodes x_k and λ_k are unknown. Consider the following cases

(1) **One-point formula** $\Rightarrow$ for $n = 0$, the (111) reduces to $\int_{-1}^{1} f(x) dx = \lambda_0 f(x_0)$

$= \frac{1}{8}\left[1 + \frac{9}{4} + \frac{9}{5} + \frac{1}{2}\right] = 0.69375$

Gauss Quadrature method $\Rightarrow$ we know that in the Integration

$$\int_a^b w(x) f(x) dx = \sum_{k=0}^{n} \lambda_k f(x_k) \quad \text{--- (1)}$$

when the nodes are also known then corresponding methods are called Newton - Cotes method, if nodes are also to be determined, then the method are called Gaussian Integration methods.

Any using the transformation $x = \left(\frac{b-a}{2}\right)t + \frac{(b+a)}{2}$

we can transformed the finite interval $[a, b]$ in the $[-1, 1]$, and the equat (1) can be transformed

$$\int_{-1}^{1} w(x) f(x) dx = \sum_{k=0}^{n} \lambda_k f(x_k)$$

$w(x) > 0$, $-1 \leq x \leq 1$ is the weight function (II)

Gauss-Legenre Integration method $\Rightarrow$

Substitute Put $w(x) = 1$ in the equation (II) we get

$$\int_{-1}^{1} f(x) dx = \sum_{k=0}^{n} \lambda_k (f(x_k)) \quad \text{--- (III)}$$

in this case all the nodes x_k and λ_k are unknown, Consider the following cases

(1) One-Point formula $\Rightarrow$ for $n = 0$, the reduces to $\int_{-1}^{1} f(x) dx = \lambda_0 f(x_0)$ (IV)

This method has two unknowns λ_0, x_0, making the method exact for $f(x) = 1, x$ we get

~~$\int_{-1}^{1} f(x)dx = \lambda_0$~~ $\int_{-1}^{1} dx = \lambda_0 f(x_0)$

$2 = \lambda_0$ $(\because f(x) = 1)$

$\int_{-1}^{1} x\,dx = \lambda_0 f(x_0)$

$\left[\frac{x^2}{2}\right]_{-1}^{1} = \lambda_0 x_0 \Rightarrow \left(\frac{1}{2} - \frac{1}{2}\right) = \lambda_0 x_0 \Rightarrow$

$\Rightarrow \lambda_0 x_0 = 0 \Rightarrow x_0 = 0$

(because $\lambda_0 = 2$)

Hence the method is given by

$\int_{-1}^{1} f(x)dx = 2f(0)$

which is the same as the mid-point formula.

The error constant is given by

$C = \int_{-1}^{1} x^2\,dx - 2(0) = \frac{2}{3}$.

$R_2 = \frac{C}{2!} f''(\xi) = \frac{1}{3} f''(\xi)$, $-1 < \xi < 1$

Similarly the two-point formula is given by

$\int_{-1}^{1} f(x)dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

The error constant is given by

$C = \int_{-1}^{1} x^4\,dx - \left(\frac{1}{9} + \frac{1}{9}\right) = \frac{2}{5} - \frac{2}{9} = \frac{8}{45}$

The error term R_4 becomes

$R_4 = \frac{C}{4!} f^{iv}(\xi) = \frac{1}{135} f^{iv}(\xi)$ $-1 < \xi < 1$

Three-point formula $n=2$, the method is given by $\int_{-1}^{1} f(x)dx = \frac{1}{9}\left(5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right)\right)$

The error constant is given by.

$$C = \int_{-1}^{1} x^6 dx - \frac{1}{9}\left[5\left(-\sqrt{\frac{3}{5}}\right)^6 + 0 + 5\left(\sqrt{\frac{3}{5}}\right)^6\right]$$

$= \frac{2}{7} - \frac{6}{25} = \frac{8}{175}$, The error in the method became $R_6 = \frac{C}{\lfloor 6} f^{(VI)}(\xi_e) = \frac{8}{\lfloor 6 \, 175} f^{(VI)}(\xi_e)$

$= \frac{1}{15750} f^{(VI)}(\xi_e)$, $-1 < \xi_e < 1$

Note: The nodes and the corresponding weights for the Gauss-Legendre Integration method $\int_{-1}^{1} f(x)dx = \sum_{k=0}^{n} \lambda_k f(x_k)$ for $n = 1$ to 5 are given by Table 5.3 on page 364 (M.K. Jain and S.R.K.)

Q-1 Evaluate the Integral $I = \int_{0}^{1} \frac{dx}{1+x}$ using Gauss-Legendre three-point formula

Solution: Transform $(0,1)$ to $(-1,1)$ by the transformation $x = \frac{b-a}{2}t + \frac{b+a}{2}$

$x = \frac{1}{2}t + \frac{1}{2} \Rightarrow \boxed{2x - 1 = t}$

here $-1 = b$, $1 = a+b$, $a = 2$, $b = -1$ and $t = 2x - 1$

$$\therefore I = \int_0^1 \frac{dx}{1+x} = \int_{-1}^{1} \frac{dt}{t+3}$$

$f(x) = \frac{1}{1+x}$

$f(t) = \frac{1}{(\frac{1}{2} + \frac{t}{2}) + 1}$

$f(t) = \frac{2}{3+t}$

using Gauss-Legendre three-point rule (Corresponding to $n=2$) we get

$$I = \frac{1}{9}\left[8\left(\frac{1}{0+3}\right) + 5\left(\frac{1}{3+\frac{\sqrt{3}}{\sqrt{5}}}\right) + 5\left(\frac{1}{3-\sqrt{3/5}}\right)\right]$$

$$= \frac{131}{189} = 0.693122$$, The exact Solution is

$$I = \log 2 \; \left[\log(1+x)\right]_0^1 = \log 2 + \log 1 = \log 2$$

$$I = 0.693147$$

Q-2 Evaluate the integral $I = \int_1^2 \frac{2x}{1+x^4}\,dx$ using the Gauss-Legendre, 1-Point, 2-point, and 3-Point Quadrature rule, compare with exact Solution

$t = \tan^{-1}(4) - \left(\frac{\pi}{4}\right)$

To use the Gauss-Legendre rules the interval $[1,2]$ is to be reduced to $[-1,1]$, so by the transformation

$$x = \left(\frac{-a+b}{2}\right)t + \frac{(a+b)}{2}$$

$$\Rightarrow x = \left(\frac{-1+2}{2}\right)t + \left(\frac{1+2}{2}\right) \Rightarrow \boxed{x = \frac{1}{2}t + \frac{3}{2}}$$

$\boxed{2x - 3 = t}$, $\therefore I = \int_{-1}^{1} 16\frac{(t+3)}{4}\,dt \Big/ 16 + (t+3)^4$

$$I = \int_{-1}^{1} \frac{8(t+3)\,dt}{16+(t+3)^4} = \int_{-1}^{1} f(t)\,dt$$

Using the One-point formula, we get

$$I = 2f(0) = 2\left(\frac{24}{16+81}\right) = 0.4948$$

Using the 2-point formula, we get

$$I = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = 0.3842 + 0.1592$$
$$= 0.5434$$

Using the 3-point rule, we get

$$I = \frac{1}{9}\left[5f\left(-\frac{\sqrt{3}}{\sqrt{5}}\right) + 8f(0) + 5f\left(\frac{\sqrt{3}}{\sqrt{5}}\right)\right]$$
$$= \frac{1}{9}\left(5(0.4343) + 8(0.2474) + 5(0.1379)\right)$$
$$= 0.5406,$$

The exact solution is $I = 0.5404$

Evalu

Q3 Evaluate the integral $I = \int_{-1}^{1} (1-x^2)^{\frac{3}{2}} \cos x\,dx$ using Gauss-Legendre 3-Point formula.

Solution ⇒ The three point formula is

$$I = \frac{1}{9}\left[5f\left(-\sqrt{\tfrac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\tfrac{3}{5}}\right)\right]$$
$$= \frac{1}{9}\left[\cancel{10\left(\tfrac{3}{5}\right)}\right.$$ here $f(x) = (1-x^2)^{\frac{3}{2}} \cos x$

$$I = \frac{1}{9}\left[10\left(\frac{3}{5}\right)^{\frac{3}{2}} \cos\left(\sqrt{\frac{3}{5}}\right) + 8\right] = 1.08979$$

Q-20 Evaluate the integrals

(I) $I = \int_0^2 \frac{dx}{3+4x}$ (II) $\int_0^2 \frac{dx}{x^2+2x+10}$

by Gauss-Legendre two-point and three point formulas

Solution (I) $I = \int_0^2 \frac{dx}{3+4x}$

~~First of all~~

we convert the finite interval [a,b] [0,2] into [-1,1] using the transformation

$x = \left(\frac{b-a}{2}\right)t + \left(\frac{b+a}{2}\right)$

$x = \left(\frac{2-0}{2}\right)t + \left(\frac{2+0}{2}\right)$

$x = t+1$

here $f(x) = \frac{1}{3+4x} \Rightarrow f(t) = \frac{1}{3+4(t+1)}$

$f(t) = \frac{1}{7+4t}$

two Point formula $\Rightarrow$ $I = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

$I = \frac{1}{7+4\left(-\frac{1}{\sqrt{3}}\right)} + \frac{1}{7+4\left(\frac{1}{\sqrt{3}}\right)}$

$I = \sqrt{3}\left[\frac{1}{7\sqrt{3}-4} + \frac{1}{7\sqrt{3}+4}\right]$

$= \left[\frac{1}{7-2.3094} + \frac{1}{7+2.3094}\right]$

$= \left[\frac{1}{4.6906} + \frac{1}{9.3094}\right] = 0.2131 + 0.1074$

$= .3305$

Three Point formula :- $f(t) = \frac{1}{7+4t}$

$$I = \frac{1}{9}\left[5 f\left(-\frac{\sqrt{3}}{\sqrt{5}}\right) + 8 f(0) + 5 f\left(\frac{\sqrt{3}}{\sqrt{5}}\right)\right]$$

$$= \frac{1}{9}\left[5\left(\frac{1}{7+4\left(-\frac{\sqrt{3}}{\sqrt{5}}\right)}\right) + \frac{8}{7} + 5\left(\frac{1}{7+4\left(+\frac{\sqrt{3}}{\sqrt{5}}\right)}\right)\right]$$

$$= \frac{1}{9}\left(5\left[\frac{1}{7-4(.77459)} + \frac{1}{7+4(.77459)}\right] + \frac{8}{7}\right]$$

$$= \frac{1}{9}\left[\left(\frac{5}{7-3.09836} + \frac{5}{7+3.09836}\right) + \frac{8}{7}\right]$$

$$\cancel{= \frac{\sqrt{5}}{9}\left[\frac{1}{3.90164} + \frac{1}{10.09836}\right] + \frac{8}{63}}$$

$$= \frac{2.2360}{9}\left[.256302478 + 0.09902598\right] + 0.1269$$

$$\cancel{= 0.03948094 + 0.1269 = 0.16638094}$$

$$= .027605(.355328458) + 0.1269$$

$$\cancel{= .009808842 + 0.1269 = 0.136708842}$$

$$= \frac{5}{9}\left[.256302478 + 0.09902598\right] + 0.1269$$

$$= 0.5555(0.355328458) + 0.1269$$

$$I = 0.324284958$$

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$$= \cancel{f(x) = \int_0^2 \frac{1}{3+4x}\,dx = \left[\frac{\log(3+4x)}{4}\right]_0^2}$$

$$\cancel{= \frac{1}{4}\left(\log(11) - \log(3)\right)}$$

$$= \frac{1}{4}\left(1.0413 - \cancel{0.477121}\right) =$$

$$\cancel{= .564271}$$

---

10. $\int_0^2 \frac{dx}{x^2+2x+10}\,dx$ by Gauss ~~legen~~ nge Legendre two-Point and three Point formulae

11) $\int_2^3 \frac{\cos 2x}{1+\sin x}\,dx$

using Gauss-Legendre two and three point integration

Composite Integration method $\Rightarrow$

(I) Trapezoidal Rule $\Rightarrow$ we divide the interval $[a, b]$ into N subinterval each of length $h = \frac{b-a}{N}$ and denote the subintervals $(x_0, x_1), (x_1, x_2) \ldots (x_{N-1}, x_N)$ where $x_0 = a$, $x_N = b$ and $x_i = x_0 + ih$, $i = 1$ to $N-1$

we have
$$I = \int_a^b f(x)\,dx$$
$$= \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \ldots \int_{x_{N-1}}^{x_N} f(x)\,dx \quad \text{— (I)}$$

evaluating each of the intervals on the right hand side of (I) by the trapezoidal rule

$$= \frac{h}{2}\left[(f_0 + f_1) + (f_1 + f_2) + \ldots + (f_{N-1} + f_N) \right]$$
$$= \frac{h}{2}\left(f_0 + 2(f_1 + f_2 + \ldots + f_{N-1}) + f_N \right) \quad \text{— (II)}$$

where $f_k = f(x_k)$ $k = 0$ to N. The formula (II) is called Composite trapezoidal rule.

(II) Simpson's Rule $\Rightarrow$ we need three abscissas for Simpson's rule of Integration. So we divide the interval $[a, b]$ into an even number of subintervals of equal length giving an odd number of abscissas. If we divide the interval $[a b]$ into $2N$ subintervals each of length $h = \frac{(b-a)}{2N}$, then we get $2N+1$ abscissas $x_0, x_1, \ldots x_{2N}$, $x_0 = a$, $x_N = b$, $x_i = x_0 + ih$

$x_i = x_0 + ih, \quad i = 1, 2, 3 \cdots 2N-1$ us

$$I = \int_a^b f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots + \int_{x_{2N-2}}^{x_{2N}} f(x)\,dx \quad \text{—(1)}$$

evaluating each of the intervals on the right hand side of (1) by the simpson's rule

$$I = \frac{h}{3}\left((f_0 + 4f_1 + f_2) + (f_2 + 4f_3 + f_4) + \cdots + f_{2N-2} + 4f_{2N-1} + f_{2N})\right)$$

$$= \frac{h}{3}\left[f_0 + 4(f_1 + f_3 + \cdots + f_{2N-1}) + 2(f_2 + f_4 + \cdots + f_{2N-2}) + f_{2N}\right] \quad \text{—(11)}$$

equation (11) is called Composite simpson's rule

Example Evaluate the integral

$$I = \int_0^1 \frac{dx}{1+x}$$

using (1) Comsite trapezoidal rule

(11) Composite Simpson's rule with $h = 2, 4,$ and 8, equal subinterval

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Soltion :⇒ when $N=2$, then $h = \frac{h}{2}$ and three nodes x_0, x_1, x_2 , $0, \frac{1}{2}, 1$, let I_T and I_S ~~represent~~ represent the value obtained by using the trapezoidal rule and Simpson's rule. we have two subintervals for trapezoidal rule. One interval for Simpson's rule.

$$~~I_S = \frac{1}{4}(f(0) + 2f(\tfrac{1}{2}))~~$$

$$I_T = \int_0^{\frac{1}{2}} f(x)\,dx + \int_{\frac{1}{2}}^{1} f(x)\,dx$$

$$= \frac{1}{2 \cdot 2}\left(f(0) + 2f(x_1) + f(x_2)\right)$$

$$I_T = \frac{1}{4}\left(1 + \frac{4}{3} + \frac{1}{2}\right) = \frac{17}{24}$$

$$I_S = \frac{1}{6}\left[f(0) + 4f(\tfrac{1}{2}) + f(1)\right] = \frac{1}{6}\left(1 + \frac{8}{3} + \frac{1}{2}\right)$$

$$= \frac{25}{36}$$

when $N = 4$ when $h = \frac{1}{4}$, five nodes $0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}$, and 1, ~~we have for~~

$$\therefore I_T = \int_0^{\frac{1}{4}} f(x) + \int_{\frac{1}{4}}^{\frac{2}{4}} f(x) + \int_{\frac{2}{4}}^{\frac{3}{4}} f(x) + \int_{\frac{3}{4}}^{1} f(x)\,dx$$

$$= \frac{1}{8}\left[f(0) + 2\left(f(\tfrac{1}{4}) + f(\tfrac{1}{2}) + f(\tfrac{3}{4})\right) + f(1)\right]$$

$$I_T = 0.697024$$

$$I_S = \int_0^{\frac{2}{4}} f(x)\,dx + \int_{\frac{2}{4}}^{1} f(x)\,dx$$

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$I_S = \frac{1}{12}[f(0) + 4f(\frac{1}{4}) + 2f(\frac{1}{2}) + 4f(\frac{3}{4}) + f(1)]$

$I_S = 0.6932254$

where $N = 8$ we have, $h = \frac{1}{8}$

$x_0 = 0$, $x_1 = \frac{1}{8}$, $x_2 = \frac{2}{8}$, ... $x_8 = 1$

we have eight subintervals for trapezoidal rule and four subinterval of Simpson's rule, we get

$I_T = \frac{1}{16}(f(0) + 2\sum_{i=1}^{7} f(\frac{i}{8}) + f(1))$

$= 0.694122$

$I_S = \frac{1}{24}(f(0) + 4\sum_{i=1}^{4} f(\frac{2i-1}{8})$

$+ 2\sum_{i=1}^{3} f(\frac{2i}{8}) + f(1)]$

$= 0.693155$,

The exact value of the integral is $I = .693147$

Do example 5.27

one page-389

MK Jain

① Evaluate $\int_{2}^{} x^3 dx$ using trapezoidal rule by dividing the range into 4 equal parts

Simpson's three-eight rule ⇒ from example 5.13 on page 355 (M.K Jain)

$$\int_{x_0}^{x_3 = x_0+3h} f(x) = \frac{3h}{8}\left(f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right) \quad (1)$$

for equally spaced point $x_i = x_0 + ih$, $i = 1, 2, 3$.

$$\int_{x_3 = x_0+3h}^{x_6 = x_0+6h} f(x) = \frac{3h}{8}\left(f(x_3) + 3f(x_4) + 3f(x_5) + f(x_6)\right]$$

,

$$\cancel{\int_{x_{3(N-1)} = x_0 + 3(N-1)h}^{x_{3N} = x_0+3Nh} f(x)\,dx = \frac{3h}{8}\left(f(x_{3(N-1)}) + 3f(x_{3N})\right)}$$

$$\int_{x_6 = x_0+3\cdot 2h}^{x_9 = x_0+3\cdot 3h} f(x) = \frac{3h}{8}\left(f(x_6) + 3f(x_7) + 3f(x_8) + f(x_9)\right)$$

$$\int_{x_{3(n-1)} = x_0 + 3(n-1)h}^{x_{3n} = x_0 + 3nh} f(x)\,dx = \frac{3h}{8}\left(f(x_{3n-3}) + 3f(x_{3n-2}) + 3f(x_{3n-1}) + f(x_{3n})\right)$$

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Similarly adding

$$\int_{x_0}^{x_0+3h} f(x)dx + \int_{x_0+3h}^{x_0+6h} f(x)dx + \int_{x_0+6h}^{x_0+9h} f(x)dx \; - - - \; \int_{x_0+3(n-3)h}^{x_0+nh} f(x)\, dx$$

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{8}\Big[f(x_0)+f(x_n) + 3\big(f(x_1)+f(x_2)+f(x_4) + f(x_5) + \cdots + f(x_{n-1})\big) + 2\big(f(x_3) + f(x_6) + f(x_{n-3})\big)\Big]$$

which is known as Simpson's three-eight rule

Q-1 Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using

(i) Trapezoidal rule

(ii) Simpson's 3/8 rule

Solution :- divide the interval (0, 6) into six parts each of width $h = 1$, the value of $f(x) = \frac{1}{1+x^2}$ are given below

x	0 x_1	1 x_2	2 x_3	3 x_4	4 x_5	5 x_6	6 x_7	$f(x) = \frac{1}{1+x^2}$
$f(x)$	1 $f(x_0)$	0.5 $f(x_1)$	0.2 $f(x_2)$	0.1 $f(x_3)$	0.0588 $f(x_4)$	0.0385 $f(x_5)$	0.027 $f(x_6)$	

① By <u>trapezoidal rule</u> ⇒

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}[f(x_0) + f(x_6) + 2(f(x_1) + f(x_2) + f(x_4) + f(x_5))]$$

$$= \frac{1}{2}[(1 + 0.027) + 2(0.5 + 0.2 + 0.1 + 0.0588 + 0.0385)]$$

$$= 1.4108$$

Ⓘ By Simpson's 3/8 Rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}(f(x_0) + f(x_6) + 3[f(x_1) + f(x_2) + f(x_4) + f(x_5)] + 2f(x_3))$$

$$= \frac{3}{8}[(1 + 0.027) + 3(0.5 + 0.2 + 0.0588 + 0.38) + 2(0.1)] = 1.3571$$

Q-3 Use the Trapezoidal rule to estimate the integral $\int_1^2 e^{x^2}\,dx$ taking the number 10 interval

Q-4 Compute the value of $\int_{0.2}^{1.4} (\sin x - \log x + e^x)\,dx$ using Simpson's $\frac{3}{8}$th rule

Q-5 Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{\sin x}$ using $\int_0^{\frac{\pi}{2}} \sin x\,dx$ by Simpson, $\frac{3}{8}$th rule

Q-6 Given that

x	4.0	4.2	4.4	4.6	4.8	5.0	5.2
$\log x$	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

evaluate $\int_4^{5.2} \log x\,dx$

(a) Trapezoidal rule (ii) Simpson, $\frac{3}{8}$th rule.

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Romberg Integration ⇒ Richardson is extrapolation procedure applied to the integration methods, is called Romber integration.
Consider the integral $I = \int_a^b f(x)\,dx$

Then the error in the composite trapezoidal rule and composite Simpson's rule can be obtain as

$$I = I_T + c_1 h^2 + c_2 h^4 + c_3 h^6 + \dots$$

$$I = I_S + d_1 h^4 + d_2 h^6 + d_3 h^8 + \dots$$

where c_i's and d_i's are constants independent of h

The extrapolation procedure for the trapezoidal rule become

$$I_T^{(m)}(h) = \frac{4^m I_T^{(m-1)}\left(\frac{h}{2}\right) - I_T^{(m-1)}(h)}{4^m - 1}, \quad m = 1, 2, \dots$$

The extrapolation procedure for the Simpson's rule become

$$I_S^{(m)}(h) = \frac{4^{m+1} I_S^{(m-1)}\left(\frac{h}{2}\right) - I_S^{(m-1)}(h)}{4^{m+1} - 1}$$

$m = 1, 2, \dots$

Example (5.28) Find the approximate value of the integral $I = \int_0^1 \frac{dx}{1+x}$

using (i) composite trapezoidal rule with 2, 3, 5, 9 nodes and Romberg integration

(1) Composite Simpson's rule with 3, 5, 9, nodes and Romberg Integration obtain,

Using the composite trapezoidal rule $\Rightarrow$

$$I = \int_a^b f(x)dx = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{N-1} f(x_i) + f(x_N)\right]$$

where $x_0 = a$, $x_N = b$, $h = \frac{b-a}{N}$, $x_i = x_0 + ih$

we get

$N = 1$, $h = 1$, $I_T = \frac{h}{2}[f(x_0) + f(x_1)] = 0.750000$

$N = 2$, $h = \frac{1}{2}$ $I_T = \frac{h}{2}[f(x_0) + 2f(x_1) + f(x_2)]$

$= 0.708333$

$N = 4$ $h = \frac{1}{4}$ $I_T = \frac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2)$
$+ 2f(x_3) + f(x_4)]$

$= 0.697024$

$N = 8$, $h = \frac{1}{8}$, $I_T = \frac{h}{2}[f(x_0) + 2\sum_{i=1}^{7} f(x_i) + f(x_8)]$

$= 0.694122$

Using Romberg integration we obtain the result

Trapezoidal Rule with Romberg Integration

h	Second order method	fourth order method	Sixth order method	Eighth order method
1	0.750000			
$\frac{1}{2}$	0.708333	0.694444		
$\frac{1}{4}$	0.697024	0.693254	0.693175	
$\frac{1}{8}$	0.694122	0.693155	0.693148	0.693148

Since exact solution is 0.693147 we required only nine function

Composite Simpson's rule ⇒

$$I = \int_a^b f(x)\,dx = \frac{h}{3}\Big[f(x_0) + 4\sum_{i=1}^{N} f(x_{2i-1}) + 2\sum_{i=1}^{N-1} f(x_{2i}) + f(x_{2N})\Big]$$

$x_0 = a,\ x_{2N} = b,\ h = \frac{b-a}{2N}$ we get

$N = 1,\ h = \frac{1-0}{2.1} = \frac{1}{2},\ I_S = \frac{h}{3}(f(x_0) + 4f(x_1) + f(x_2))$

$= 0.694444$

$N = 2,\ h = \frac{1}{4}\quad I_S = \frac{h}{3}[f(x_0) + 4(f(x_1) + f(x_3)) + f(x_4)]$

$I_S = 0.693254$

$N = 4\quad h = \frac{1}{8}\ ;\ I_S = \frac{h}{3}[f(x_0) + 4(f(x_1) + f(x_3) + f(x_5) + f(x_7)) + 2(f(x_2) + f(x_4) + f(x_6)) + f(x_8)]$

$= 0.693155$

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Using Romberg integration we obtain the result

h	Fourth order method	Sixth order method	Eight order method
$\frac{1}{2}$	0.694444		
$\frac{1}{4}$	0.693254	0.693175	
$\frac{1}{8}$	0.693155	0.693148	0.693148

$$I_T^{(m)}(h) = \frac{4^m I_T^{(m-1)}(\frac{h}{2}) - I_T^{(m-1)}(h)}{4^m - 1}, \quad m = 1, 2 \ldots$$

where $I_T^{(0)}(h) = I_T(h)$

The computed result is of order $O(h^{2m+2})$

The extrapolation using three ~~step~~ step lengths $h, \frac{h}{2}, \frac{h}{2^2}$ are given

Romberg method of trapezoidal Rule

Step length	Value of I $O(h^2)$	Value of I $O(h^4)$	Value of I $O(h^6)$
h	$I(h)$		
		$I^{(1)}(h) = \frac{4I(\frac{h}{2}) - I(h)}{3}$	
$\frac{h}{2}$	$I(\frac{h}{2})$		$I^{(2)}(h) = \frac{16 I^{(1)}(\frac{h}{2}) - I^{(1)}(h)}{15}$
		$I^{(1)}(\frac{h}{2}) = \frac{4I(\frac{h}{4}) - I(\frac{h}{2})}{3}$	
$\frac{h}{4}$	$I(\frac{h}{4})$		

Note that most accurate value are values at the end each column

Romber method for Simpson's $\frac{1}{3}$ rule

$$I_S^{(m)}(h) = \frac{4^{m+1} I_S^{(m-1)}\left(\frac{h}{2}\right) - I_S^{(m-1)}(h)}{4^{m+1} - 1}$$

where $m = 1, 2.$

$I_S^{(0)}(h) = I_S(h)$, the computed result is of order $O(h^{2m+4})$

the extrapolation using three step length $h, \frac{h}{2}, \frac{h}{2^2}$ are given

Romberg method for Simpson's $\frac{1}{3}$ rule

Step length	Value of I $O(h^4)$	Value of I $O(h^6)$	Value of I $O(h^8)$
h	$I(h)$		
		$I^{(1)}(h) = \frac{16\left(\frac{h}{2}\right) - I(h)}{15}$	
$\frac{h}{2}$	$I\left(\frac{h}{2}\right)$		$I^{(2)}(h) = \frac{64\, I^{(1)}\left(\frac{h}{2}\right) - I^{(1)}(h)}{63}$
		$I^{(1)}\left(\frac{h}{2}\right) = \frac{16\left(I\left(\frac{h}{4}\right)\right) - I\left(\frac{h}{2}\right)}{15}$	
$\frac{h}{4}$	$I\left(\frac{h}{4}\right)$		

Note that the most accurate values are the values at the end of each column

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Simpson's $\frac{1}{3}$ Rule $\Rightarrow$ Let the interval $[a, b]$ be subdivided into two equal parts with step length $h = \left(\frac{b-a}{2}\right)$ where we have three abscissae $x_0 = a$, $x_1 = \frac{a+b}{2}$, $x_2 = b$

we know that $\Rightarrow$

$$I = \int_a^b f(x)\,dx = \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

Q-1 find the approximate value of $I = \int_0^1 \frac{dx}{1+x}$ using the Simpson's $\frac{1}{3}$ rule with 2, 4, 8 equal subintervals

Solution $\Rightarrow$ $N = 1$ $h = \frac{b-a}{2N} = \frac{1}{2}$ Then Nodes $0, 0.5, 1.0$

$N = 2$ $h = \frac{b-a}{2N} = \frac{1}{4}$ the nodes are $0, 0.25, 0.5, 0.75, 1.0$

$N = 4$ $\Rightarrow$ $h = \frac{b-a}{2N} = \frac{1}{8}$, the nodes are $0, 0.125, 0.25, 0.375, 0.5, 0.625, 0.75, 0.875, 1.$

we have the following table

$n = 2N = 2$
$N = 1$

x	0	0.5	1.0
$f(x)$	1.0	0.666667	0.5

$n = 2N = 4$
$N = 2$

x	0	0.125	0.25	0.375	0.5	0.625	.75	0.875
$f(x)$	1							

x	0	0.25	0.5	0.75	1.0
$f(x)$	1.0	.8	0.666667	.571429	0.5

1

$n = 2N = 8$
$N = 4$

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
$f(x)$	1	0.88889	0.8	.727273	0.666667	0.615385	0.571429		

Now compute interval integral

$n = 2N = 2$

$h = \frac{1}{2}$ $\quad I_1 = \frac{h}{3}(f(0) + 4f(0.5) + f(1.0))$

$= \frac{1}{6}(1.0 + 4(0.666667) + 0.5)$

$= 0.674444$

$n = 2N = 4$

$I_2 = \frac{h}{3}\left(f(0) + 4(f(0.25) + f(0.75)) + 2f(0.5) + f(1.0)\right)$

$= \frac{1}{12}(1.0 + 4(0.8 + 0.571429) + 2(0.666667) + 0.5) = 0.643254$

$n = 2N = 8$ $\quad I_3 = \frac{h}{3}(f(0) + f(0.125) + f(0.375) + f(0.625) + f(0.875))$

$h = \frac{1}{8}$ $\quad + 2\{f(0.25) + f(0.5) + f(1.0)\}]$

~~$I_3 = \frac{h}{3}[$~~

$= \frac{1}{24}[1.0 + 4\{0.888889 + 0.727273 +$
$0.615385 + 0.533333\}$
$+ 2\{0.8 + 0.666667 + 0.5714294\} + 0.5]$

$= 0.693155$

The exact value of the integral is

$I = \log 2 = 0.693147$

Q-2 Evaluate $I = \int_1^2 \frac{dx}{5+3x}$ using the Simpson's 1/3 rule with 4 and 8 subintervals. Compare with the exact solution and ~~find the absolute error in the solution~~

Q-3 Using Simpson's $\frac{1}{3}$ rule evaluate the integral $I = \int_0^1 \frac{dx}{x^2+6x+10}$ with h · 2 and 4 subinterval. Compare with exact solution

Q-4 Evaluate $I = \int_1^2 \frac{dx}{5+3x}$ with 4 and 8 subinterval using the trapezoidal rule. Compare with exact solution.

Q-14 Calculate $\int_0^{\frac{1}{2}} \frac{x}{\sin x}\, dx$

(i) using the open type formulas (5.78), (5.79) and (5.80)

~~(ii) using the semi-open type formulas~~

(ii) using the trapezoidal rule with $h = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ and Romberg integration. Assume $f(0)$ is taken as the limiting value.

Solution ⇒ (i) mid-point rule $n=2$, $x_0 = 0$, $x_n = \frac{1}{2}$

here $h = \frac{b-a}{n} = \frac{\frac{1}{2}-0}{2} = \frac{\frac{1}{2}}{2} = \frac{1}{4}$

$$\therefore \int_0^{\frac{1}{2}} \frac{x}{\sin x}\, dx = 2h f(x_0+h) = \frac{2}{4}\left(\frac{(x_0+\frac{1}{4})}{\sin(x_0+\frac{1}{4})}\right)$$

$$= \frac{1}{2}\frac{\left(\frac{1}{4}\right)}{\sin\left(\frac{1}{4}\right)} = \frac{1}{8\sin(0.25)} = 0.505247$$

(ii) Three-point formula ($n=3$), $x_0 = 0$, x_0+h, x_0+2h, $x_0+3b = b$, $h = \frac{\frac{1}{2}-0}{3} = \frac{1}{6}$

$$\therefore \int_a^b f(x)\,dx = \frac{3h}{2}\left(f(x_0+h) + f(x_0+2h)\right) \cancel{+ 2 f(x_0+3h)}$$

$$\therefore \int_0^{\frac{1}{2}} \frac{x}{\sin x}\,dx \qquad + \frac{14h^5}{45} f^{IV}(\xi_3)$$

$$= \frac{3}{2}\left(\frac{1}{6}\right)\left[f\left(\frac{1}{6}\right) + f\left(\frac{2}{6}\right)\right]$$

$$= \frac{1}{4}\left[f\left(\frac{1}{6}\right) + f\left(\frac{2}{6}\right)\right] = 0.505852$$

(iii) Three point rule ($n=4$): $x_0=0$, x_0+h, x_0+2h, x_0+3h, $x_0+4h=\frac{1}{2}$, $h=\frac{\frac{1}{2}-0}{4}=\frac{1}{8}$

$$\therefore \int_a^b f(x)dx = \frac{4h}{3}\left(2f(x_0+h)-f(x_0+2h)+2f(x_0+3h)\right)$$

$$\therefore \int_0^{\frac{1}{2}} \frac{x}{\sin x} = \frac{4}{3}\left(\frac{1}{8}\right)\left[2f\left(\frac{1}{8}\right)-f\left(\frac{4}{8}\right)+2f\left(\frac{3}{8}\right)\right)$$

$$= 0.507064.$$

(ii) Using trapezoidal rule with $h=\frac{1}{2},\frac{1}{4},\frac{1}{8}$, and Romber integration. Assum $f(0)$ is taken as the limite value (ii) $\lim_{x\to 0} f(x)=f(0)$

$$\Rightarrow f(0)=\lim_{x\to 0}\left(\frac{x}{\sin x}\right)$$

$$=\lim_{x\to 0}\left(\frac{1}{\cos x}\right)\left(\frac{0}{0}\right)$$

$$=1$$

$$\boxed{f(0)=1}$$

For $h=\frac{1}{2}$ $x_0=0$, $x_1=0+\frac{1}{2}=b=\frac{1}{2}$

$$\therefore I_T = \frac{h}{2}\left(f(x_0)+f(x_1)\right)=\frac{1}{4}\left(f(0)+f\left(\frac{1}{2}\right)\right)$$

$$=\frac{1}{4}\left(1+\frac{\frac{1}{2}}{\sin\left(\frac{1}{2}\right)}\right)=0.510729$$

For $h=\frac{1}{4}$, $x_0=0$, $x_1=0+\frac{1}{4}$, $x_2=0+\frac{2}{4}=\frac{b}{2}$

$$I_T=\frac{h}{2}\left[f(x_0)+2f(x_1)+f(x_2)\right)$$

$$=\frac{1}{8}\left(1+2\frac{\left(\frac{1}{4}\right)}{\sin\left(\frac{1}{4}\right)}+\frac{\frac{1}{2}}{\sin\frac{1}{2}}\right)=0.507988$$

for $\frac{h}{8} = \frac{1}{8}$, $x_0 = 0$, $x_1 = 0 + \frac{1}{8}$, $x_2 = 0 + \frac{2}{8}$, $x_3 = \frac{3}{8}$

$x_4 = 0 + \frac{4}{8} = \frac{1}{2} = b$

$$I_T = \frac{h}{2}\left(f(0) + 2\left\{ f\left(\tfrac{1}{8}\right) + f\left(\tfrac{2}{8}\right) + f\left(\tfrac{3}{8}\right) \right\} + f\left(\tfrac{1}{2}\right) \right)$$

$$= \frac{1}{16}\left[1 + 2\left\{ \left(\frac{\frac{1}{8}}{\sin\frac{1}{8}}\right) + \left(\frac{\frac{2}{8}}{\sin\frac{2}{8}}\right) + \left(\frac{\frac{3}{8}}{\sin\frac{3}{8}}\right) \right\} + \left(\frac{\frac{1}{2}}{\sin\frac{1}{2}}\right) \right]$$

$= 0.507298$

Trapezoidal Rule with Romberg integration

h	Second order method	Fourth order	Sixth order	Eight order method
$\frac{h}{2}$	0.510729			
		$= .507074333$		
$\frac{h}{4}$	0.507988	$\frac{4 \times 0.507988 - .510729}{3}$		
			.50767578	
$\frac{h}{8}$	0.507298			
		.507068		

Extrapolated Value $I = 0.507068$

Q-15 Compute $\int_{\frac{\pi}{4}}^{\frac{3\pi}{2}} \left(\frac{1}{\sin x}\right)^{\frac{1}{4}} dx$ using the open type formula (5.79) and (5.80)

Q-16 $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x \log(\sin x)}{\sin^2 x + 1}\, dx$ Correct to 3 decimal places

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(i) using trapezoidal rule and Romberg integration
(ii) using Simpson's rule and Romberg integration

Solution :- Using Composite trapezoidal rule ⇒

$$I = \int_a^b f(x)\,dx = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{N-1} f(x_i) + f(x_N)\right]$$

where $x_0 = \frac{\pi}{4}$, $x_N = \frac{\pi}{2}$, $h = \dfrac{\frac{\pi}{2} - \frac{\pi}{4}}{N}$, $x_i = x_0 + ih$

For $N = 1$, $h = \dfrac{\frac{\pi}{2} - \frac{\pi}{4}}{1} = \frac{\pi}{4}$,

$$I_T = \frac{\pi}{4\cdot 2}\left[f\left(\frac{\pi}{4}\right) + f\left(\frac{\pi}{2}\right)\right] = -.064158$$

For $N = 2$, $h = \dfrac{\frac{\pi}{2} - \frac{\pi}{4}}{2} = \frac{\pi}{8}$, $x_i = x_0 + ih$

$$I_T = \frac{\pi}{8\cdot 2}\left[f(x_0) + 2f(x_1) + f(x_2)\right]$$

$$= \frac{\pi}{8}\left[f\left(\frac{\pi}{4}\right) + 2f\left(\frac{\pi}{4} + \frac{\pi}{8}\right) + f\left(\frac{\pi}{4} + \frac{2\pi}{8}\right)\right]$$

$$= \frac{\pi}{16}\left[f\left(\frac{\pi}{4}\right) + 2f\left(\frac{3\pi}{8}\right) + f\left(\frac{\pi}{2}\right)\right]$$

$I_T = -0.038498$

For $N = 4$, $h = \dfrac{\frac{\pi}{2} - \frac{\pi}{4}}{4} = \frac{\pi}{4\cdot 4} = \frac{\pi}{16}$

$$I_T = \frac{\pi}{16\cdot 2}\left[f\left(\frac{\pi}{4}\right) + 2f\left(\frac{\pi}{4} + \frac{\pi}{16}\right) + 2f\left(\frac{\pi}{4} + \frac{2\pi}{16}\right) + 2f\left(\frac{\pi}{4} \ldots\right. + f\left(\frac{\pi}{2}\right)$$

$= -0.031531$

Trapezoidal Rule with Romberg integration

h	Second order method $O(h^2)$	fourth order method $O(h^4)$	Sixth order method $O(h^6)$
$\frac{\pi}{4}$	-0.064158		
		-0.0299446	
$\frac{\pi}{8}$	-0.038498		-0.29159
		-0.0292086	
$\frac{\pi}{16}$	-0.031531		

~~using~~ Similarly using the composite Simpson's rule

$$I = \int_a^b f(x)dx = \frac{h}{3}\left[f(x_0) + 4\sum_{i=1}^{N} f(x_{2i-1}) + 2\sum_{i=1}^{N-1} f(x_{2i}) + f(x_{2N})\right]$$

here $x_0 = \frac{\pi}{4}$ $x_{2N} = \frac{\pi}{2}$ $h = \frac{b-a}{2N}$ we get

for $N = 1$, $h = \frac{\frac{\pi}{2} - \frac{\pi}{4}}{2\cdot 1} = \frac{\pi}{8}$

$$I_8 = \frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right)$$

$$= \frac{\pi}{8\cdot 3}\left[f\left(\frac{\pi}{4}\right) + 4f\left(\frac{\pi}{4} + \frac{\pi}{8}\right) + f\left(\frac{\pi}{4} + \frac{2\pi}{8}\right)\right]$$

$$= \frac{\pi}{24}\left[f\left(\frac{\pi}{4}\right) + 4f\left(\frac{3\pi}{8}\right) + f\left(\frac{\pi}{2}\right)\right]$$

$$= -0.029945$$

For $N=2$ $h = \frac{\frac{\pi}{2}-\frac{\pi}{4}}{2\cdot 2} = \frac{\frac{\pi}{4}}{4} = \frac{\pi}{16}$

$I_S = \frac{h}{3}\left(f(x_0) + 4(f(x_1) + f(x_3)) + 2f(x_2) + f(x_4)\right)$

$= \frac{\pi}{48}\left(f\left(\frac{\pi}{4}\right) + 4\left(f\left(\frac{\pi}{4}+\frac{\pi}{16}\right) + f\left(\frac{\pi}{4}+\frac{3\pi}{16}\right)\right)\right.$
$\left. + 2f\left(\frac{\pi}{4}+\frac{2\pi}{16}\right) + f\left(\frac{\pi}{4}+\frac{4\pi}{16}\right)\right)$

$= \frac{\pi}{48}\left(f\left(\frac{\pi}{2}\right) + 4\left(f\left(\frac{5\pi}{16}\right) + f\left(\frac{8\pi}{16}\right)\right)\right.$
$\left. + 2 f\left(\frac{6\pi}{16}\right) + f\left(\frac{\pi}{2}\right)\right) = -0.029209$

Simpson's Rule with Romberg Integration

h	fourth order method $O(h^4)$	Sixth order method $O(h^6)$	
$\frac{\pi}{8}$	-0.029945		
$\frac{\pi}{16}$	-0.029209	-0.029160	

① $\frac{Q-17}{p-348}$ (M.K Jain), (19)