→ If $f(x)$ is a bounded f^n in $[a,b]$, then if P is any partition of $[a,b]$; $P = a = x_0 < x_1 < \cdots < x_n = b$, then i_{th} subinterval of partition is taken as.
$\Delta x_i = [x_{i-1}, x_i]$ & it is also used to indicate the length of i_{th} subinterval.

$$\Delta x_i = [x_{i-1}, x_i] = x_i - x_{i-1}$$

Max. Δx_i is know as Norm of partition of P
or
Mesh.

$$\mu(P) = \max_{P} \Delta x_i$$

* If f^n $f(x)$ is defined in $[a,b]$ then let m & M be inf. & sup (in set of real no.) of $f(x)$ in $[a,b]$

let m_i & M_i be infimum & supremum of $f(x)$ in i_{th} subinterval $[x_{i-1}, x_i]$

infimum of $f(x)$ in $[a,b] \leq$ infimum of $f(x)$ in $[x_{i-1}, x_i]$

i.e. $m \leq m_i \leq M_i \leq M$

* If $f(x)$ is bounded f^n in $[a,b]$, then corresponding to any partition P upper sum of the f^n is given by

$$\boxed{U(f,a,b) = \sum_{i=1}^{n} \underbrace{M_i}_{\text{supremum in } i_{th} \text{ interval}} \Delta x_i} \rightarrow \text{length of } i_{th} \text{ subinterval}$$

also known as Upper Darbox sum.

Similarly, lower sum of f^n

$$L(f,a,b) = \sum_{i=1}^{n} m_i \Delta x_i \rightarrow \text{lower Darbox sum.}$$

* Corresponding any partition P, multiply by Δx_i

$$m \Delta x_i \leq m_i \Delta x_i \leq M_i \Delta x_i \leq M \Delta x_i$$

$$\sum_{i=1}^{n} m \Delta x_i \leq \sum_{i=1}^{n} m_i \Delta x_i \leq \sum_{i=1}^{n} M_i \Delta x_i \leq \sum_{i=1}^{n} M \Delta x_i$$

$$m(b-a) \leq L(P,a,b) \leq U(P,a,b) \leq M(b-a)$$

Ques: Consider $f(x) = \begin{cases} 1 & ; x \in \mathbb{Q} \\ -1 & ; x \in \mathbb{Q}^c \end{cases}$, $[0,1]$, find Upper & lower Darbox corresponding to any in the partition P.

Sol^n: i-th subinterval $[x_{i-1}, x_i]$ $<$ infinite $\mathbb{Q}$, infinite $\mathbb{Q}^c$

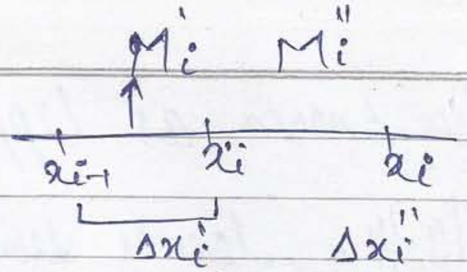
$M_i = 1$, $m_i = -1$

$$U_f(P) = \sum_{i=1}^{n} 1 \cdot \Delta x_i = 1(b-a) = 1 \cdot 1 = 1$$

$$L_f(P) = \sum_{i=1}^{n} (-1) \Delta x_i = -1(b-a) = -1.$$

1/11/2013

$$U(P, f) = \sum M_i \Delta x_i$$
$$L(P, f) = \sum m_i \Delta x_i$$



$$M_i', M_i'' \leq M_i$$

$$M_i' \Delta x_i' + M_i'' \Delta x_i'' \leq M_i(\Delta x_i' + \Delta x_i'')$$
$$\leq M_i \Delta x_i$$

$$m_i' \Delta x_i' + m_i'' \Delta x_i'' \geq m_i(\Delta x_i' + \Delta x_i'')$$
$$\geq m_i \Delta x_i$$

* the seq^n of upper sum monotonically decrease if we add new points into the given interval to Partition it into subintervals. & it is bounded below, so it will converges to its infimum

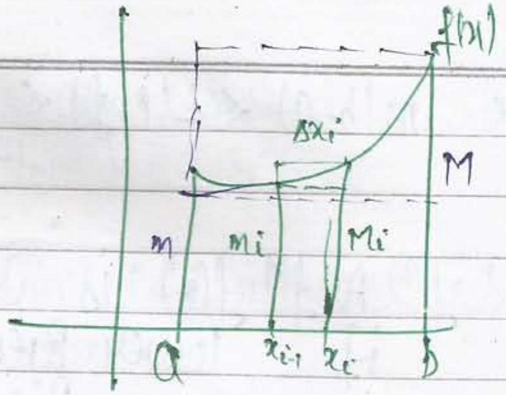
* the seq^n of lower sum in such case will be monotonically increasing & it is bounded above so it will converges to its supremum

* If P_1 & P_2 are any two partition, then lower sum $L(P_1, f) \leq U(P_2, f)$

let us partition $P = P_1 \cup P_2$

$$L(P_1, f) \leq L(P_1 \cup P_2, f) \leq U(P_1 \cup P_2, f) \leq U(P_2, f)$$
$$\Rightarrow L(P_1, f) \leq U(P_2, f)$$

$$\sup_{P} L(P, f) \leq \inf_{P} U(P, f)$$



* $\int_a^b f(x)\, dx.$

Refinement of Partition :

If P is any partition, then we add one or more points into it, then ~~we~~ the new partition is refinement of the previous partition. denoted by P^*

$$L(P^*, f) \geq L(P, f)$$
$$U(P^*, f) \leq U(P, f)$$

Def^n: Infimum of the upper Darbox sum taken over all the possible partition is known as upper Riemann Integral of the f^n in the interval $[a, b]$ & it is denoted by.

$$\text{Upper Riemann Integral} = \overline{\int_a^b} f(x)\, dx.$$

Similarly, Supremum of lower sum taken over all the partition is known as lower Riemann Integral of the f^n from a to b.

$$\text{Lower Riemann Integral} = \underline{\int_a^b} f(x)\, dx$$

* $m(b-a) \le L(P,f) \le \underline{\int_a^b} f(x)dx \le \overline{\int_a^b} f(x)dx \le U(P,f) \le M(b,a)$

Defⁿ: A f^n $f(x)$ is s.t.b Riemann integrable in $[a,b]$ if lower Riemann integral of f^n in that interval = Upper Riemann integral " " " "

$$\boxed{L \cdot R \cdot I = U \cdot R \cdot I}$$

& the ~~commo~~ common value is known as integral of the f^n in $[a,b]$

$$L \cdot R \cdot I = U \cdot R \cdot I = \int_a^b f(x)\,dx.$$

(this is Riemann's condⁿ for integrability.)

Ques: Consider a f^n $f(x) = \begin{cases} 1 & ; x \in \mathbb{Q} \\ -1 & ; x \in \mathbb{Q}^c \end{cases}$. Is this f^n is Riemann integrable in $[a,b]$

Solⁿ $U(P,f) = \sum 1 \cdot \Delta x_i = (b-a)$

$L(P,f) = \sum(-1)\Delta x_i = (a-b) = -(b-a)$

$\sup L(P,f) = -(b-a) = \overline{J}$

$\inf U(P,f) = b-a = \underline{J}$

$L \cdot R \cdot I = U \cdot R \cdot I$ if $b-a = 0$

$a = b$ (Not possible)

$\Rightarrow$ the given f^n is not Riemann integrable in any interval.

Ques Consider $f(x) = K$, where K is any constant f^n. Is $f(x)$ is Riemann integrable in $[a,b]$

Sol^n: Corresponding to any partition in i_{th} interval. Infimum & supremum is same

$$\inf U(P,f) = \sum K\Delta x_i = K(b-a)$$
$$\sup L(P,f) = \sum K\Delta x_i = K(b-a)$$

$\Rightarrow$ the given f^n is Riemann integrable & the value of integral is common value of U.R.I & L.R.I $= \int_a^b f(x)\,dx$.

Thm: If a f^n $f(x)$ is Riemman integrable in $[a,b]$, then $\forall\ \epsilon > 0$, $\exists\ \delta > 0$ s.t.

$$I - L(P,f) < \epsilon \text{ whenever } \underset{\text{norm}}{\mu(P)} < \delta$$

where I = integral.

(2) If $f(x)$ is Riemann integrable f^n, then $\forall\ \epsilon > 0$, $\exists\ \delta > 0$ s.t.

$$U(P,f) - I < \epsilon \text{ whenever } \mu(P) < \delta.$$

Oscillatory sum:

If corresponding to partition P, $M_i - m_i$ is known as oscillation of $f(x)$ in the i_{th} - subinterval.

Oscillatory sum of i_{th} - interval $= (M_i - m_i)\Delta x_i$ or Total sum of i_{th} - interval

Oscillatory sum in interval $[a,b]$ corresponding to taken partition P

$$= \sum_{i=1}^{n} (M_i - m_i)\, \Delta x_i$$

$$= U(P,f) - L(P,f)$$

& denoted by

$$\boxed{\omega(P,f) = U(P,f) - L(P,f)}$$

Thm : A necessary & sufficient condⁿ of a fⁿ $f(x)$ to be Riemann integrable in $[a,b]$ is that $\forall\, \epsilon > 0,\ \exists\, \delta > 0$ s.t.

$U(P,f) - L(P,f) < \epsilon$ whenever $\mu(P) < \delta$.

Proof: A fⁿ $f(x)$ is Riemann integrable

$\forall\, \epsilon/2 > 0,\ \exists\, \delta_1 > 0$ s.t.

$U(P,f) - I < \epsilon/2$ whenever $\mu(P) < \delta_1$

$(\delta_2 > 0)$ $\quad I - L(P,f) < \epsilon/2$ " $\mu(P) < \delta_2$

$U(P,f) - L(P,f) < \epsilon$ whenever $\mu(P) < \min\{\delta_1, \delta_2\}$

$\Rightarrow \mu(P) < \delta$"

Conversely,

$U(P,f) - L(P,f) < \epsilon$ whenever $\mu(P) < \delta$

$\overline{J} - \underline{J} \le U - L < \epsilon$

$\Rightarrow \overline{\int} f(x)\,dx - \underline{\int} f(x)\,dx < \epsilon$ whenever $\mu(P) < \delta$

if $\epsilon \to 0$

$\overline{\int} f(x)\,dx - \underline{\int} f(x)\,dx \to 0$

$$\overline{\int} f(x)\,dx = \underline{\int} f(x)\,dx$$

$\Rightarrow f(x)$ is Riemann integral.

$$\Rightarrow \lim_{\mu(P)\to 0} \omega(P,f) = 0$$

$$\lim_{\mu(P)\to 0} U(P,f) - L(P,f) = 0$$

Thm ① If $f(x)$ is Riemann integrable f^n, then $Kf(x)$ is also Riemann integrable f^n.

Proof $U(P,f) - L(P,f) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i < \epsilon/_{\delta}$

whenever $\mu(P) < \delta$

$$\sum_{i=1}^{n} (K)(M_i - m_i)\Delta x_i < \cancel{\epsilon/_{\delta}} \cdot \delta = \epsilon$$

② If f_1 & f_2 are R.I f^n in $[a,b]$, then $f_1 + f_2$ is also R.I f^n in that interval.

$$\sup(f_1+f_2) \leq \sup f_1 + \sup f_2$$
$$\inf(f_1+f_2) \geq \inf f_1 + \inf f_2$$

inf f+g, sup f + sup g

inf f_1 + inf g, sup f+g

$$\sup(f_1+f_2) - \inf(f_1+f_2) \leq \left((M_i^{f_1} - m_i^{f_1}) + (M_i^{f_2} - m_i^{f_2})\right)$$

$$\sum\left[\sup(f_1+f_2) - \inf(f_1+f_2)\right]\Delta x_i \leq \sum\left((M_i^{f_1} - m_i^{f_1}) + (M_i^{f_2} - m_i^{f_2})\right)\cdot\Delta x_i$$

$$U(P, f_1+f_2) - L(P, f_1+f_2) < \epsilon/2 + \epsilon/2 = \epsilon$$

$\mu(P) < \delta_1$; $\mu(P) < \delta_2$

$\Rightarrow \mu(P) < \delta$

where $\delta = \min\{\delta_1, \delta_2\}$

$\Rightarrow f_1 + f_2$ is Riemann integrable.

Cor: ① If f & g are Riemann integrable, then $af + bg$ is R.I

② if f & g are Riemann integrable in $[a,b]$, then $f \cdot g$ is also Riemann integrable in that interval.

* If M_i^f, M_i^g, m_i^f & m_i^g are supremum & infimum of f & g resp, then

$$M_i^{f+g} - m_i^{f+g} \leq |M_i^f M_i^g - m_i^f m_i^g|$$

* If $f(x)$ is R.I f^n in $[a,b]$, then $|f|$ is also R.I in that interval.

M_i, m_i – supremum & infimum of $|f|$

$M_i', m_i' \rightarrow$ ~~inf~~ sup & " of f.

$$M_i - m_i \leq (M_i' - m_i')$$

$$\sum (M_i - m_i)\Delta x_i < \sum \underbrace{(M_i' - m_i')\Delta x_i}_{R.I} \longrightarrow 0$$

$$\Rightarrow \sum (M_i - m_i)\Delta x_i \longrightarrow 0$$

$\omega(P, |f|) \to 0$

$\Rightarrow |f|$ is also R·I.

Is this Converse True ??

$$f(x) = \begin{cases} 1 & ; x \in \mathbb{Q} \\ -1 & ; x \in \mathbb{Q}^c \end{cases}$$

$|f(x)| = 1 \; ; x \in \mathbb{R}.$

Thm Que: If $f(x)$ is R·I f^n, then $f^2(x)$ is also R·I.

Yes, $f^2(x) = \underbrace{f(x) f(x)}_{\text{product of two } f^n}$

$\Rightarrow f^2(x)$ is R·I

Converse of above thm need not be true.

$$f(x) = \begin{cases} 1 & ; x \in \mathbb{Q} \\ -1 & ; x \in \mathbb{Q}^c \end{cases}$$

$$f^2(x) = \begin{cases} 1 & ; x \in \mathbb{Q} \\ 1 & ; x \in \mathbb{Q}^c \end{cases} \Rightarrow 1 \; ; x \in \mathbb{R}.$$

Thm: If $f(x)$ is R·I f^n in $[a,b]$ & $\exists M > 0$ s.t. $|f(x)| \geq M \;\; \forall x \in [a,b]$

then $\frac{1}{f(x)}$ is also R·I in that interval.

Sol^n Let M_i & m_i are supremum & infimum of $f(x)$

$\Rightarrow 1/m_i$ & $1/M_i$ are " " of $1/f(x)$

$U^f < U^g$
$L^f < L^g$

$U^f - L^f < U^g - L^g$
$\cancel{U^g - U^f} \geq 0$

$$\left|\frac{1}{m_i} - \frac{1}{M_i}\right| = \left|\frac{M_i - m_i}{M_i m_i}\right| \leq \frac{M_i - m_i}{M^2}$$

$$\Rightarrow \sum_{i=1}^{n} \left|\frac{1}{m_i} - \frac{1}{M_i}\right| \Delta x_i \leq \sum_{i=1}^{n} \frac{(M_i - m_i)}{M_2} \Delta x_i$$

$$< \frac{\epsilon M^2}{M_2} = \epsilon \quad \forall\ \epsilon M^2 > 0$$

whenever $\mu(P) < \delta$

$\Rightarrow \frac{1}{f(x)}$ is R.I.

Thm: if $f(x)$ & $g(x)$ are R.I f^n in the interval $[a,b]$ & $|g(x)| \geq \lambda > 0\ \forall\ x \in [a,b]$, then $f(x)/g(x)$ is also R.I in that interval.

If $f(x)$ & $g(x)$ are two f^n's in $[a,b]$ s.t. $f(x) \leq g(x)\ \forall\ x \in [a,b]$, then corresponding to any partition P which of the following is true.

(A) $U(P,g) - L(P,g) \leq U(P,f) - L(P,f)$
(B) " $\geq$ "
(C) " $=$ "
(D) $U(P,g) - \cancel{L(P,f)}\ L(P,f) \geq U(P,f) - L(P,g)$

$m_i^f \leq m_i^g$
$M_i^f \leq M_i^g$

$L(P,f)$ $L(P,g)$

$U(P,f)$ $U(P,g)$

Thm: If $f(x)$ is conts in $[a,b]$, then $f(x)$ is R·I in $[a,b]$

Proof: Conts in $[a,b] \Rightarrow$ Uniformly conts in $[a,b]$

As, $f(x)$ is uniformly conts in $[a,b]$
$\forall\ \delta_1 > 0\ \exists\ \delta_2 > 0$ s.t.
$|f(x_1) - f(x_2)| < \delta_1$ whenever $|x_1 - x_2| < \delta_2$.

for any conts f^n, in any interval (closed), infimum & supremum is a member of the f^n values

$$\sum (M_i - m_i)\Delta x_i = \sum (f(x_1) - f(x_2))\Delta x_i$$

Choose δ_1 s.t. $\delta_1 < \frac{\epsilon}{(b-a)}$

$$\sum [f(x_1) - f(x_2)]\Delta x_i < \sum \frac{\epsilon}{(b-a)}\Delta x_i$$

$$< \frac{\epsilon\cancel{(b-a)}}{\cancel{(b-a)}} = \epsilon$$

$\Rightarrow \sum (M_i - m_i)\Delta x_i < \epsilon$ whenever $\mu(P) < \delta_2$

(Converse of above thm need not be true

If a f^n $f(x)$ is Monotonic in $[a,b]$, then it is also R·I in $[a,b]$

Proof: $\sum (M_i - m_i)\Delta x_i = \sum [f(x_i) - f(x_{i-1})]\Delta x_i$

$\Delta x_i < \frac{\epsilon}{[f(b) - f(a) + 1]}$

length of

$\sum(M_i - m_i)\,\Delta x_i \leq \frac{\epsilon}{(f(b)-f(a)+1)} \sum_{i=1}^{n} f(x_i) - f(x_{i-1})$

$\leq \frac{\epsilon}{(f(b)-f(a)+1)} \cdot (f(b)-f(a))$

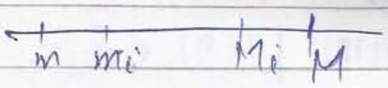
$\sum(M_i - m_i)\Delta x_i < \epsilon$, whenever $\mu(P) < \frac{\epsilon}{f(b)-f(a)+1}$

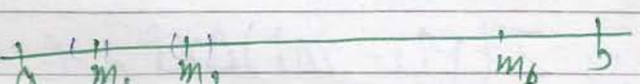
$\Rightarrow f(x)$ is R.I

* $\int_0^{50} [x]\,dx$, $[x]$ is monotonically increasing
$\Rightarrow [x]$ is R.I
But $[x]$ is not conts.

* A f^n $f(x)$ is R.I if set of points of discontinuity of $f(x)$ in $[a,b]$ is finite.

Let $m_1, m_2, \ldots m_k$ be k points of discontinuities of $f(x)$ in $[a,b]$





Oscillation of $f(x)$ in any subinterval $\leq (M-m)$

To Enclose the k point of discontinuities is non-overlapping subinterval of length $\frac{\epsilon}{2k(M-m)}$

& In the remaining $(k+1)$ subinterval, $f(x)$ is conts

& Hence, R.I

$\forall \ \frac{\epsilon}{2(k+1)} > 0, \ \exists \ \delta > 0$ s.t.

oscillatory sum in each subinterval $< \frac{\epsilon}{2(k+1)}$; $\mu(P) < \delta$

$$\omega(P, f) < (k+1)\frac{\epsilon}{2(k+1)} + (M-m) \times \frac{\epsilon}{2k(M-m)} \times k$$

$$= \epsilon$$

$$\omega(P, f) < \epsilon \quad ; \ \mu(P) < \min.\left\{\delta, \frac{\epsilon}{2k(M-m)}\right\}$$

If set of points of discontinuities of $f(x)$ has finite no. of limit points in $[a,b]$, then $f(x)$ is Riemann integrable in $[a,b]$.

Proof: Let $m_1, m_2, \ldots, m_k$ be the limit points of points of discontinuities of $f(x)$ in $[a,b]$, then
then enclose each limits points within the Non-lapping subinterval of length $< \frac{\epsilon}{2k(M-m)}$

Oscillation in any subinterval can't exceed by $M-m$
then, in the ~~rem~~ remaining $(k+1)$ subinterval, there will be only finite point discontinuity.
(By Bolzana - ~~Weier~~ W. thm)
in the remaining $(k+1)$ sub-interval, f^n will by R.o.I

$\forall \ \frac{\epsilon}{2(k+1)} > 0, \ \exists \ \delta > 0$ s.t

$$O.S < \frac{\epsilon}{2(k+1)} \ ; \ \mu(P) < \delta$$

$$\omega(P,f) < (K+1)\frac{\epsilon}{2(K+1)} + (M-m)\frac{\epsilon}{2K(M-m)}K$$

$$\omega(P,f) < \epsilon, \text{ whenever } \mu(P) < \min\left\{\delta, \frac{\epsilon}{2k(M-m)}\right\}$$

Consider of f^n $f(x) = \begin{cases} 1 - \frac{1}{2^k}; & \text{when } x \in \left[\frac{1}{2^k}, \frac{1}{2^{k+1}}\right] \\ & k = 1, 2, \dots \\ 0 & ; x = 0 \end{cases}$

$$f(x) = \begin{cases} 1 - \frac{1}{2} & ; x \in \left(\frac{1}{2}, 1\right] \\ 1 - \frac{1}{2^2} & ; x \in \left(\frac{1}{2^2}, \frac{1}{2}\right] \\ 1 - \frac{1}{2^3} & ; x \in \left(\frac{1}{2^3}, \frac{1}{2}\right] \\ \vdots & \\ 0 & ; x = 0 \end{cases}$$

$f(x)$ is discontinuous at $\left\{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots\right\} = S$

Limit point of $S = \{0\}$ (By previous thm)

$\Rightarrow f(x)$ is R.I.

Thm: If $f(x)$ is f^n, which is derivative of $F(x)$ $\left[\text{i.e. } \frac{d}{dx}F(x) = f(x)\right]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$