Piecewise linear Interpolation

* Let $x_0, x_1, x_2, \dots, x_n$ be $n+1$ distinct nodal points
the piecewise <u>linear interpolating</u> polynomial

$$P_{i,1}(x) = \frac{x - x_i}{x_{i-1} - x_i} f(x_{i-1}) + \frac{x - x_{i-1}}{x_i - x_{i-1}} f(x_i) \;, \quad i = 1, 2, \dots, n$$

where $x \in [x_{i-1}, x_i]$

<u>Ex</u>: Obtain the piecewise linear interpolating polynomials for the function $f(x)$ defined by the data

x	1	2	4	8
$f(x)$	3	7	21	73

find values of $f(3)$ and $f(7)$

<u>Sol^n</u>: $x_0 = 1$, $f(x_0) = 3$
$x_1 = 2$, $f(x_1) = 7$
$x_2 = 4$, $f(x_2) = 21$
$x_3 = 8$, $f(x_3) = 73$

$$P_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

(Here $i=1$)

where $x \in [x_0, x_1]$, $P_1(x)$ denote as linear interpolation
i.e $x \in [1, 2]$

$$P_1(x) = \frac{x-2}{(1-2)}(3) + \frac{(x-1)}{(2-1)} 7$$

$$\Rightarrow -3(x-2) + 7(x-1)$$

$$\Rightarrow 4x - 1$$

Now, $i = 2$

$$\Rightarrow x \in [x_1, x_2] \Rightarrow x \in [2, 4]$$

$$P_1(x) = \frac{x-4}{(2-4)}(7) + \frac{(x-2)}{(4-2)} 21$$

$\Rightarrow 7x - 7$

Now $i = 3$

$x \in [4, 8]$

$$P_1(x) = \frac{x-8}{(4-8)}(21) + \frac{(x-4)}{(8-4)}(73)$$

$\Rightarrow 13x - 31$

the piecewise linear interpolating polynomials are given as.

$$P_1(x) = \begin{cases} 4x-1 & , 1 \le x \le 2 \\ 7x-7 & , 2 \le x \le 4 \\ 13x-31 & , 4 \le x \le 8. \end{cases}$$

(1) $f(3) = ?$

$3 \in [2, 4]$

$\Rightarrow P_1(x) = 7x - 7$

$f(3) = P_1(3) = 7(3) - 7$

$= 14.$

(2) $f(7) = ?$

$7 \in [4, 8]$

$\Rightarrow P_1(x) = 13x - 31$

$= 13(7) - 31$

$\Rightarrow 91 - 31$

$\Rightarrow 60$

Piecewise Quadratic Interpolation

let $[a,b]$ be given interval
we subdivide the given interval $[a,b]$
where $a = x_0, x_1, \ldots, x_{2n} = b$
~~Given~~ ~~Subinterval~~ On each of the subintervals,

$$\underbrace{\begin{matrix} x_0 & x_1 & x_2 \\ f(x_0) & f(x_1) & f(x_2) \end{matrix}}_{\text{Ist Interval}} \quad \underbrace{\begin{matrix} x_3 & x_4 \\ f(x_3) & f(x_4) \end{matrix}}_{\text{2nd Interval}} \quad \underbrace{\begin{matrix} x_{2n-2} & x_{2n} \\ f(x_{2n-1}) & f(x_{2n}) \end{matrix}}_{2n^{th}\ \text{Interval}}$$

The Quadratic Interpolating polynomial.

$$f_2(x) = \frac{(x-x_i)(x-x_{i+1})}{(x_{i-1}-x_i)(x_{i-1}-x_{i+1})} f(x_{i-1}) + \frac{(x-x_{i-1})(x-x_{i+1})}{(x_i-x_{i-1})(x_i-x_{i+1})} f(x_i)$$

$$+ \frac{(x-x_{i-1})(x-x_i)}{(x_{i+1}-x_{i-1})(x_{i+1}-x_i)} f(x_{i+1}) \quad \text{---} \circledast$$

Ex Obtain the piecewise quadratic interpolating polynomials for the function $f(x)$ defined by the data.

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
x	-3	-2	-1	1	3	6	7
$f(x)$	369	222	171	165	207	990	1779

Hence, find an approximate value of $f(-2.5)$ & $f(6.5)$

Soln Here Ist Interval $= [x_0, x_2] = [-3, -1]$
IInd Interval $= [x_2, x_4] = [-1, 3]$
IIIrd Interval $= [x_4, x_6] = [3, 7]$

let $i = 1$
from $\circledast$.

quadratic eq^n

$$P_{1,2}(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1)$$

Ist Interval

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

Nodal points $\{-3,-2,-1\}$

$$\Rightarrow \frac{(x+2)(x+1)}{(-3+2)(-3+1)} 369 + \frac{(x+3)(x+1)}{(-2+3)(-2+1)} (222)$$

$$+ \frac{(x+3)(x+2)}{(-1+3)(-1+2)} (171)$$

$$= \frac{(x^2+3x+2)\,369}{2} + \frac{(x^2+4x+3)\,222}{1} + \frac{171(x^2+5x+6)}{2}$$

$$\Rightarrow$$

$$= 48x^2 + 93x + 1548$$

Here $i = 2$, $\{\overset{x_1}{-1}, \overset{x_2}{1}, \overset{x_3}{3}\}$ – point.

$$P_{(2,2)}(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} f(x_1) + \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} f(x_2)$$

$$+ \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} f(x_3)$$

$$\Rightarrow \frac{(x-1)(x-3)}{(-1-1)(-1-3)} (171) + \frac{(x+1)(x-3)}{(1+1)(1-3)} (165)$$

$$+ \frac{(x+1)(x-1)}{(3+1)(3-1)} (207)$$

$$\Rightarrow 6x^2 - 3x + 162$$

points = $\{3, 6, 7\}$

$$\underset{\substack{\downarrow \\ \text{3rd} \\ \text{interval}}}{P_{3,2}}(x) = \frac{(x-6)(x-7)}{(3-6)(3-7)} \cdot (207) + \frac{(x-3)(x-7)}{(6-3)(6-7)}(990) + \frac{(x-3)(x-6)}{(7-3)(7-6)}(1779)$$

$$= 132x^2 - 927x + 1800$$

Qus: Obtain the piecewise linear interpolating polynomial for the function $f(x)$ defined by the given data.

x	0	1	2	3
$f(x)$	1	2	5	10

4 interpolate at $x = 0.5$ and 1.5.