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Lecture - 13, 14

INTERPOLATION

Suppose, we are given the following values of $y = f(x)$, for a set values of x:

x :	x_0	x_1	x_2	...	x_n
y :	y_0	y_1	y_2	...	y_n

Then, the process of finding the values of y corresponding to any value of $x = x_i$ between x_0 and x_n, is called Interpolation.

That is,

interpolation estimates the value of function between any intermediate value of the variable x.

★ The process of computing the value of function outside the given range is called Extrapolation.

The interpolation is based on the calculus of finite differences. Two important formulaes are:

① Newton's Forward Interpolation Formula.

② Newton's Backward Interpolation Formula.

Newton's Forward Interpolation Formula

Let, the function $y = f(x)$ takes the values $y_0, y_1, \ldots, y_n$ corresponding to $x_0, x_1, \ldots, x_n$ of x.

let, $x_i = x_0 + i \cdot h \quad (i = 0, 1, \ldots)$

Suppose $y(x)$ be a polynomial of degree n.
We can write,

$$y(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

From there, we find that

$$a_0 = y_0 \quad , \quad \Delta y_0 = y_2 - y_1 = a_1 . h .$$

$$\Rightarrow \quad a_1 = \frac{1}{h} \cdot \Delta y_0 .$$

and $\Delta y_1 = y_2 - y_1 = \Delta y_0 + 2h^2 a_2$

$$\Rightarrow \quad a_2 = \frac{1}{2!h^2} \Delta^2 y_0$$

Similarly, $a_3 = \dfrac{1}{3!h^3} \Delta^3 y_0$.

therefore,

$$a_0 = y_0, \quad a_1 = \frac{1}{h}\Delta y_0, \quad a_2 = \frac{1}{2!h^2}\Delta^2 y_0, \quad \dots, \quad a_n = \frac{1}{n!h^n}\Delta^n y_0 .$$

Now, suppose we want to evaluate y for x
where, $x = x_0 + ph$, then

$$x - x_0 = ph \quad \Rightarrow \quad p = \frac{x - x_0}{h}$$

and $\boxed{\begin{aligned} x - x_1 &= (p-1)h \\ x - x_2 &= (p-2)h \\ &\vdots \\ x - x_n &= (p-n)h. \end{aligned}}$

Hence, $y(x) = y(x_0 + ph) = y_p$ is given by

$$\boxed{y_p = y_0 + p \cdot \Delta y_0 + \frac{p(p-1)}{2!} \cdot \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \dots + \frac{p(p-1)\dots(p-(n-1))}{n!} \Delta^n y_0}$$

where, $\Delta^k y_{m-1} = \Delta^{k-1} y_m - \Delta^{k-1} y_{m-1}$

that is, $\Delta y_{m-1} = y_m - y_{m-1}$

and $\Delta^2 y_{m-1} = \Delta y_m - \Delta y_{m-1}$

Eg: The table gives the distance in nautical miles of the visible horizon for the given height in feet above the earth's surface:

height $= x$:	100	150	200	250	300	350	400
distance $= y$:	10.63	13.03	15.04	16.81	18.42	19.90	21.27

find : The value of y, when

(i) $x = 160$ ft

(ii) $x = 410$ ft

x	y	Δ	Δ^2	Δ^3	Δ^4
100	10.63				
		2.40			
150	13.03		−0.39		
		2.01		0.15	
200	15.04		−0.24		−0.07
		1.77		0.08	
250	16.81		−0.16		−0.05
		1.61		0.03	
300	18.42		−0.13		−0.01
		1.48		0.02	
350	19.90		−0.11		
		1.37			
400	21.27				

(1) For, $x_0 = 160$, $y_0 = 13.03$ $\{\because 150 < 160\}$

therefore,

$$\Delta y_0 = y_1 - y_0$$
$$= 15.04 - 13.03$$
$$= 2.01$$

and $$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$$
$$= 1.77 - 2.01$$
$$= -0.24$$

and $$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$$
$$= -0.16 + 0.24$$
$$= 0.08$$

and $$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$$
$$= 0.03 - 0.08$$
$$= -0.05$$

Since, $x = 160$ and $h = 50$,

$\therefore \quad p = \frac{x - x_0}{h} = \frac{10}{50} = 0.2$

$\therefore$ Using Newton's forward interpolation formula, we get :-

$$y_{160} = y_0 + p.\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p.(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \dots$$

$$= 13.03 + 0.2(2.01) + \frac{(0.2)(-0.8)(-0.24)}{2!} + \frac{(0.2)(-0.8)(-1.8)(0.08)}{3!} + \frac{(0.2)(-0.8)(-1.8)(-2.8)(-0.05)}{4!}$$

$$= 13.03 + 0.402 + 0.192 + 0.0384 + 0.00168$$

$$= 13.46.$$

Newton's Backward Interpolation Formula

Let the function $y = f(x)$ takes the values $y_0, y_1, y_2, \dots$ corresponding to $x_0, x_1, x_2, \dots$

Suppose, it is required to evaluate $f(x)$ for $x = x_n + ph$, where $p \in \mathbb{R}$.

Then,

$y_p = y(x_n + ph) = y(x)$ is given by

$$y_p = y_n + p.\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \dots$$