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Laplace Transformation

Introduction

Laplace transforms help in solving the differential equations with boundary values without finding the general solution and the values of the arbitrary constants.

Laplace transform

Definition: let $f(t)$ be function defined for all positive values of t, then

$$F(s) = \int_0^{\infty} e^{-st} f(t)\,dt$$

provided the integral exists, is called the Laplace Transform of $f(t)$, it is denoted as

$$L(f(t)) = F(s) = \int_0^{\infty} e^{-st} f(t)\,dt$$

Important formulae

(1) $L(1) = \frac{1}{s}$

(2) $L(t^n) = \frac{n!}{s^{n+1}}$, where $n = 0, 1, 2, 3 \ldots$

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3) $L(e^{at}) = \frac{1}{s-a}$ $(s>a)$

4.) $L(\cosh at) = \frac{s}{s^2-a^2}$ $(s^2>a^2)$

5) $L(\sinh at) = \frac{a}{s^2-a^2}$ $(s^2>a^2)$

6) $L\ \sin(at) = \frac{a}{s^2+a^2}$ $(s>0)$

7) $L(\cos at) = \frac{s}{s^2+a^2}$ $(s>0)$

1. $L(1) = \frac{1}{s}$

Proof: $L(1) = \int_0^\infty 1 \cdot e^{-st} dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty$

$= \frac{-1}{s}\left[\frac{1}{e^{st}}\right]_0^\infty$

$= \frac{-1}{s}\left[0-1\right] = \frac{1}{s}$

Hence $L(1) = \frac{1}{s}$ Proved

2. Hence $L(t^n) = \frac{n!}{s^{n+1}}$, where n and s are positive

$$L(t^n) = \int_0^\infty e^{-st} t^n dt$$

Putting $st = x$ or $t = \frac{x}{s}$ or $dt = \frac{dx}{s}$

thus, we have $L(t^n) = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s}$

or $L(t^n) = \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n dx$

or $L(t^n) = \frac{n!}{s^{n+1}}$ $\left[\Gamma n+1 = \int_0^\infty e^{-x} x^n dx \text{ and } \Gamma n+1 = n! \right]$

Hence Prooved

3. $L(e^{at}) = \frac{1}{s-a}$ where $s > a$

Proof. $L(e^{at}) = \int_0^\infty e^{-st} \cdot e^{at} dt = \int_0^\infty e^{-st+at} \cdot dt$

$$= \int_0^\infty e^{(-s+a)t} dt = \int_0^\infty e^{-(s-a)t} dt.$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty$$

$$= \frac{-1}{s-a}\left[\frac{1}{e^{(s-a)t}}\right]_0^{\infty}$$

$$= \frac{-1}{(s-a)}(0-1) = \frac{1}{s-a}$$

4. $L(\cosh at) = \frac{s}{s^2-a^2}$

Proof: $L(\cosh at) = L\left[\frac{e^{at}+e^{-at}}{2}\right]$

$$= \frac{1}{2}L(e^{at}) + \frac{1}{2}L[e^{-at}]$$

$\left[\because \cosh at = \frac{e^{at}+e^{-at}}{2}\right]$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$\left[L(e^{at}) = \frac{1}{s-a}\right]$

$$= \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right] = \frac{s}{s^2-a^2}$$

5. $L(\sinh at) = \frac{a}{s^2-a^2}$

Proof $L(\sinh at) = L\left[\frac{1}{2}\left(e^{at} - e^{-at}\right)\right]$

$$= \frac{1}{2}\left[L(e^{at}) - L(e^{-at})\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2}\right]$$

$$= \frac{a}{s^2-a^2}$$

Prooved

6. $L(\sin at) = \frac{a}{s^2+a^2}$

Proof. $L(\sin at) = L\left[\frac{e^{iat} - e^{-iat}}{2i}\right]$

$$= \frac{1}{2i}\left[L\left(e^{iat} - e^{-iat}\right)\right] = \frac{1}{2i}\left[L(e^{iat}) - L(e^{-iat})\right] \quad \left[\because \sin at = \frac{e^{iat} - e^{-iat}}{2i}\right]$$

$$= \frac{1}{2i}\left[\frac{1}{s-ia} - \frac{1}{s+ia}\right] = \frac{1}{2i}\,\frac{s+ia-s+ia}{s^2+a^2}$$

$$= \frac{1}{2i}\,\frac{2ia}{s^2+a^2} = \frac{a}{s^2+a^2}$$

Proved

Questions

Q1: - Find the Laplace transform of f(t) defined as

$$f(t) = \frac{t}{k}, \text{ when } 0 < t < k$$
$$= 1, \text{ when } t > k$$

Solution:

$$f(t) = \int_0^k \frac{t}{k} e^{-st} dt + \int_k^\infty 1 \cdot e^{-st} dt$$

$$= \frac{1}{k}\left[\left(\frac{te^{-st}}{-s}\right)_0^k - \int_0^k \frac{e^{-st}}{-s} dt\right] + \left[\frac{e^{-st}}{-s}\right]_k^\infty$$

$$= \frac{1}{k}\left[\frac{ke^{-ks}}{-s} - \left(\frac{e^{-st}}{s^2}\right)_0^k\right] + \frac{e^{-ks}}{s}$$

$$= \frac{1}{k}\left[\frac{ke^{-ks}}{-s} - \frac{e^{-sk}}{s^2} + \frac{1}{s^2}\right] + \frac{e^{-ks}}{s}$$

$$= \frac{-e^{-sk}}{s} - \frac{1}{k}\frac{e^{-ks}}{s^2} + \frac{1}{k}\frac{1}{s^2} + \frac{e^{-ks}}{s}$$

$$= \frac{1}{ks^2}\left[1 - e^{-ks}\right]$$

Answer

Q2: From the first principle, find the laplace transform of $(1+\cos 2t)$

$$\Rightarrow \int_0^\infty e^{-st}(1+\cos 2t)\,dt$$

$$= \int_0^\infty e^{-st}\left(1+\frac{e^{2it}+e^{-2it}}{2}\right)dt$$

$$= \frac{1}{2}\int_0^\infty \left[2e^{-st}+e^{(-s+2i)t}+e^{(-s-2i)t}\right]dt$$

$$= \frac{1}{2}\left[\frac{2e^{-st}}{-s}+\frac{e^{(-s+2i)t}}{-s+2i}+\frac{e^{(-s-2i)t}}{-s-2i}\right]_0^\infty$$

$$= \frac{1}{2}\left[\left(0+\frac{2}{s}\right)+\frac{1}{-s+2i}(0-1)+\frac{1}{-s-2i}(0-1)\right]$$

$$= \frac{1}{2}\left[\frac{2}{s}+\frac{1}{s-2i}+\frac{1}{s+2i}\right]$$

$$= \frac{1}{2}\left[\frac{2}{s}+\frac{2s}{s^2+4}\right]$$

$$= \frac{1}{s}+\frac{s}{s^2+4} = \frac{2s^2+4}{s(s^2+4)}$$

Properties of Laplace Transforms :-

(i) $L[af_1(t) + bf_2(t)] = aL(f_1(t)) + bL[f_2(t)]$

Proof. $L(af_1(t) + bf_2(t)] = \int_0^\infty e^{-st}[af_1(t) + bf_2(t)]$

$= a\int_0^\infty e^{-st} f_1(t) + b\int_0^\infty e^{-st} f_2(t)\,dt$

$= aLf_1(t) + bLf_2(t)$

(2) First shifting Theorem

if $Lf(t) = F(s)$, then

$L(e^{at}f(t)) = F(s-a)$

Proof :-

$L[e^{at}f(t)] = \int_0^\infty e^{-st}.e^{at} f(t)\,dt$

$= \int_0^\infty e^{-(s-a)t} f(t)\,dt$

$= \int_0^\infty e^{-rt} f(t)\,dt$

where $r = s-a$)

$= F(r) = F(s-a)$

with the help of this property, we can have the following important results:

(1) $L(e^{at} t^n) = \frac{n!}{(s-a)^{n+1}}$ $\left[L(t^n) = \frac{n!}{s^{n+1}} \right]$

(2) $L(e^{at} \cosh bt) = \frac{s-a}{(s-a)^2 - b^2}$

(3) $L(e^{at} \sinh bt) = \frac{b}{(s-a)^2 - b^2}$

(4) $L(e^{at} \sin bt) = \frac{b}{(s-a)^2 + b^2}$

(5) $L(e^{at} \cos bt) = \frac{s-a}{(s-a)^2 + b^2}$

questions

example 3:- Find the laplace transform of $\cos^2 t$.

Ans :-

$\cos 2t = 2\cos^2 t - 1$

$\cos^2 t = \frac{1}{2}\left[\cos 2t + 1\right]$

$L(\cos^2 t) = L\left[\frac{1}{2}(\cos 2t + 1)\right]$

$= \frac{1}{2}\left[L(\cos 2t) + L(1)\right]$

$= \frac{1}{2}\left[\frac{s}{s^2+4} + \frac{1}{s}\right] = \frac{1}{2}\left[\frac{s}{s^2+4} + \frac{1}{s}\right]$

example 4. Find the Laplace Transform of $t^{-1/2}$

Ans: we know that $L(t^n) = \frac{\Gamma n+1}{s^{n+1}}$

Put $n = -\frac{1}{2}$, $L(t^{-1/2})$

$$= \frac{\Gamma -\frac{1}{2}+1}{s^{-\frac{1}{2}+1}} = \frac{\Gamma \frac{1}{2}}{\sqrt{s}} = \frac{\sqrt{\pi}}{\sqrt{s}}$$

where $\Gamma \frac{1}{2} = \sqrt{\pi}$

example 5. Find the laplace transform of $t \sin at$.

$$L(t \sin at) = L\left[t \frac{e^{iat} - e^{-iat}}{2i}\right]$$

$$= \frac{1}{2i}\left[L(t \cdot e^{iat}) - L(t e^{-iat})\right]$$

$$= \frac{1}{2i}\left[\frac{1}{(s-ia)^2} - \frac{1}{(s+ia)^2}\right]$$

$$= \frac{1}{2i}\left[\frac{(s+ia)^2-(s-ia)^2}{(s-ia)^2(s+ia)^2}\right]$$

$$= \frac{1}{2i}\left[\frac{(s^2+2ias-a^2)-(s^2-2ias-a^2)}{(s^2+a^2)^2}\right]$$

$$= \frac{4ias}{2i(s^2+a^2)^2} = \frac{2as}{(s^2+a^2)^2}$$

example 6. Find the laplace Transform of $t^2 \cos at$.

$$L(t^2\cos at) = L\left[t^2 \cdot \frac{e^{iat}+e^{-iat}}{2}\right]$$

$$= \frac{1}{2}\left[L(t^2 e^{iat}) + L(t^2 e^{-iat})\right]$$

$$= \frac{1}{2}\left[\frac{\lfloor 2}{(s-ia)^3} + \frac{\lfloor 2}{(s+ia)^3}\right]$$

$$= \frac{(s+ia)^3+(s-ia)^3}{(s-ia)^3(s+ia)^3}$$

$$= \frac{(s^3+3ias^2-3a^2s-ia^3)+(s^3-3ias^2-3a^2s+ia^3)}{(s^2+a^2)^3}$$

$$= \frac{2s^3 - 6a^2 s}{(s^2+a^2)^3} = \frac{2s(s^2 - 3a^2)}{(s^2+a^2)^3} \text{ Ans}$$

"Laplace Transform of the Derivative of $f(t)$"

$$L[f'(t)] = sL[f(t)] - f(0). \text{ where } L[f(t)] = F(s)$$

Proof: $L[f'(t)] = \int_0^\infty e^{-st} f'(t)\, dt$

Integerating by parts, we get

$$L[f'(t)] = \left[e^{-st} \cdot f(t)\right]_0^\infty - \int_0^\infty (-se^{-st}$$

$$= -f(0) + s\int_0^\infty e^{-st} f(t)\, dt$$

$$\left[e^{-st} f(t) = 0, \text{ where } t = \infty\right]$$

$$= -f(0) + sLf(t).$$

$$L[f'(t)] = sLf(t) - f(0) \quad \text{— (A)}$$

Laplace transform of derivative of $f(t)$ correspon to multiplication of the Laplace transform of $f(t)$ by s.

"Laplace transform of derivative of order n.

$$L[f^{n}(t)] = s^{n} L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \cdots f^{n-1}(0)$$

using (A)

$$L[f'(t)] = s\,L f(t) - f(0).$$

$\boxed{f(t)\,dt}$

Replacing $f(t)$ by $f'(t)$ and $f'(t)$ by $f''(t)$ in (1) we get

$$L[f''(t)] = s\,L[f'(t)] - f'(0)$$

Putting the value of $L[f'(t)]$ from (1) in (2), we have

$$L[f''(t)] = s[s\,L[f(t)] - f(0)] - f'(0)$$

Or $$L[f''(t)] = s^2 L[f(t)] - s f(0) - f'(0).$$

Ily $$L[f'''(t)] = s^3 L[f(t)] - s^2 f(0) - s f'(0) - f''(0)$$

$$L[f^{iv}(t)] = s^4 L[f(t)] - s^3 f(0) - s^2 f'(0) - s f''(0) - f'''(0)$$

$$L[f^n(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \cdots - f^{n-1}(0)$$

★ Laplace Transform of integral of $f(t)$:

$$L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s} f(s)$$

where $L[f(t)] = f(s)$

Proof:- Let $\phi(t) = \int_0^t f(t)\,dt$

and $\phi(0) = 0$, then $\phi'(t) = f(t)$

We know the formula of Laplace Transforms of $\phi'(t)$ i.e

$$L[\phi'(t)] = sL[\phi(t)] - \phi(0)$$

$$L[\phi'(t)] = sL[\phi(t)]$$

$$L[\phi(t)] = \frac{1}{s} L[\phi'(t)]$$

Putting the values of $\phi(t)$ and $\phi'(t)$ we get

$$L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s} L[f(t)] \text{ or}$$

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$L\left[\int_0^{\infty} f(t)\, dt\right] = \frac{1}{s} F(s)$ Proved

Note
(i) Laplace Transform of integral of $f(t)$ corresponding to the division of the Laplace transform of $f(t)$ by s.

(2) $\int_0^t f(t) = L^{-1}\left[\frac{1}{s} F(s)\right]$

Laplace Transform of $t f(t)$ (Multiplication by t)

$L[f(t)] = F(s)$, then

$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)]$

Proof:- $L[f(t)] = F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$

Differentiating (1) w.r.t "s", we get

$\frac{d}{ds}[F(s)] = \frac{d}{ds}\left[\int_0^{\infty} e^{-st} f(t)\, dt\right]$

$= \int_0^{\infty} \frac{\partial}{\partial s} e^{-st} f(t)\, dt$

$$= \int_0^\infty (-t e^{-st}) \cdot f(t)\, dt$$

$$= \int_0^\infty e^{-st} [-t \cdot f(t)]\, dt$$

$$= L[-t f(t)] \text{ or } L[t f(t)]$$

$$= (-1)^1 \frac{d}{ds}[F(s)]$$

Similarly $L[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2}[F(s)]$

$$L[t^3 f(t)] = (-1)^3 \frac{d^3}{ds^3}[F(s)]$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)]$$

Proved

Q. Find the laplace transform of $t \sinh at$.

Sol:-

$$L(\sinh at) = \frac{a}{s^2 - a^2}$$

$$\therefore \quad L[t \sinh at] = -\frac{d}{ds}\left(\frac{a}{s^2 - a^2}\right)$$

or $\quad L[t \sinh at] = \frac{2as}{(s^2 - a^2)^2}$

Q:- Find the Laplace transform of $t^2 \cos at$

$L(\cos at) = \frac{s}{s^2+a^2}$

$L(t^2 \cos at) = (-1)^2 \frac{d^2}{ds^2}\left[\frac{s}{s^2+a^2}\right]$

$= \frac{d}{ds} \frac{(s^2+a^2)\cdot 1 - s(2s)}{(s^2+a^2)^2}$

$= \frac{d}{ds} \frac{a^2-s^2}{(s^2+a^2)^2}$

$= \frac{(s^2+a^2)^2(-2s) - (a^2-s^2)\cdot 2(s^2+a^2)(2s)}{(s^2+a^2)^4}$

$= \frac{-2s^3 - 2a^2 s - 4a^2 s + 4s^3}{(s^2+a^2)^3}$

$= \frac{2s(s^2-3a^3)}{(s^2+a^2)^3}$ Ans.

Q:- Obtain the laplace transform of $t^2 e^t \sin 4t$.

Solution:

$L(\sin 4t) = \dfrac{4}{s^2+16}$, $L(e^t \sin 4t) = \dfrac{4}{(s-1)^2+16}$

$L(t\, e^t \sin 4t) = -\dfrac{d}{ds}\, \dfrac{4}{s^2-2s+17}$

$= \dfrac{-4(2s-2)}{(s^2-2s+17)^2}$

$L(t^2 e^t \sin 4t) = -4 \dfrac{d}{ds} \dfrac{2s-2}{(s^2-2s+17)^2}$

$= \dfrac{-4(s^2-2s+17)^2 - (2s-2)\, 2(s^2-2s+17)(2s-2)}{(s^2-2s+17)^4}$

$= \dfrac{-4(2s^2-4s+34-8s^2+16s-8)}{(s^2-2s+17)^3}$

$= \dfrac{-4(-6s^2+12s+26)}{(s^2-2s+17)^3}$

$= \dfrac{8(3s^2-6s-13)}{(s^2-2s+17)^3}$ Ans.