Thm CAUCHY'S ROOT TEST

Let ΣU_n be a series of +ve terms.

let $l = \lim_{n\to\infty} U_n^{1/n}$

if $l<1$, then ΣU_n Converges

If $l>1$, then " diverges

if $l=1$, " " may converge or diverge.

Proof: case (i) $l<1$

choose $\epsilon>0$ s.t. $l+\epsilon<1$

As $\lim_{n\to\infty} U_n^{1/n} = l$

$\therefore$ for each $\epsilon>0$, $\exists\, n_0 \in \mathbb{N}$ s.t.

$$l-\epsilon < U_n^{1/n} < l+\epsilon \quad \forall\, n \geq n_0$$

This gives $U_n < (l+\epsilon)^n \quad \forall\, n\geq n_0$

Now, $\Sigma (l+\epsilon)^n$ is a Geom. series with $r = l+\epsilon<1$

$\Rightarrow$ Converges

$\therefore$ By Basic Comp. test

ΣU_n also converges.

Case(ii) (a) $1<l<\infty$ (i.e $l>1$, but finite)

choose $\epsilon>0$ s.t. $l-\epsilon>1$

As $\lim_{n\to\infty} U_n^{1/n} = l$

$\therefore$ for each $\epsilon>0$, $\exists\, n_0\in\mathbb{N}$ s.t.

$$l-\epsilon < U_n^{1/n} < l+\epsilon \quad \forall\, n\geq n_0$$

This gives, $U_n > (l-\epsilon)^n \quad \forall\, n\geq n_0$

Now, $\Sigma(l-\epsilon)^n$ is a Geom. series with $r = l-\epsilon>1$

and hence diverges

$\therefore$ By Basic Comp. test, ΣU_n also diverges.

Case (ii) (b) $l = \infty$

Choose any $k > 1$

As $\lim_{n\to\infty} u_n^{1/n} = \infty$ $\therefore$ for each $k > 0$, $\exists\ n_0 \in \mathbb{N}$ s.t.

$$u_n^{1/n} > k \quad \forall\ n \geq n_0$$

This gives: $u_n > k^n \quad \forall\ n \geq n_0$

Now Σk^n is a geom. series with $r = k > 1$ and hence diverges

$\therefore$ By Basic Comp. test,

Σu_n also diverges.

Case (iii) $l = 1$

first consider the series $\Sigma 1/n^2$ (Convergent as $p = 2 > 1$)

Here $u_n = 1/n^2$

$$l = \lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \left(1/n^2\right)^{1/n} = \lim_{n\to\infty} \left(1/n^{1/n}\right)^2 = 1$$

$\left(\because \lim_{n\to\infty} n^{1/n} = 1\right)$

Next consider the series $\Sigma 1/n$ (divergent as $p = 1$)

Here $u_n = 1/n$

$$l = \lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} (1/n)^{1/n} = \lim_{n\to\infty} 1/n^{1/n} = 1/1 = 1$$

Thus, series for which $l = 1$ may converge or diverge.

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Eg: Test for convergence.

$$\frac{2}{3}+\left(\frac{3}{5}\right)^2+\left(\frac{4}{7}\right)^3+\cdots$$

Sol^n $U_n = \left(\frac{n+1}{2n+1}\right)^n$

$$l = \lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \frac{n+1}{2n+1} = \frac{1}{2} < 1$$

$\therefore$ By Root Test,

ΣU_n converges.

Eg 2. Test for convergence $\Sigma\left(n^{1/n}-1\right)^n$

Sol^n

$$U_n = \left(n^{1/n}-1\right)^n$$

$$\lim_{n\to\infty} U_n^{1/n} = \lim_{n\to\infty}\left(n^{1/n}-1\right) = 1-1 = 0 < 1$$

By Root Test.

ΣU_n converges.

Eg: Test for convergence $\Sigma\left(\frac{n}{n-1}\right)^{n^2}$

Sol^n. $U_n = \left(\frac{n}{n-1}\right)^{n^2}$

$$\lim_{n\to\infty} U_n^{1/n} = \lim_{n\to\infty}\left(\frac{n}{n-1}\right)^n = \lim_{n\to\infty}\frac{n^n}{n^n\left(1-1/n\right)^n}$$

$$= \lim_{n\to\infty}\frac{1}{\left(1-1/n\right)^n} = \frac{1}{e^{-1}} = e > 1$$

$\therefore$ By Root Test,

ΣU_n diverges.

Thm:

D'ALEMBERT'S RATIO TEST

let ΣU_n be a series of +ve terms

let $l = \lim_{n\to\infty} \frac{U_n}{U_{n+1}}$

if $l>1$, then ΣU_n converges

if $l<1$, " " diverges

if $l=1$, " " may converge or diverge.

Proof: Case (i)(a) $l>1$ but finite (ie $1<l<\infty$)

choose $\epsilon>0$ s.t. $l-\epsilon>1$

As $\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = l$ $\therefore$ $\epsilon>0$, $\exists$ $n_0 \in \mathbb{N}$ s.t.

$$l-\epsilon < \frac{U_n}{U_{n+1}} < l+\epsilon \quad \forall\ n \geq n_0$$

In particular, taking $n = n_0, n_0+1, n_0+2, \ldots$ we get.

$$\left.\begin{aligned} \frac{U_{n_0}}{U_{n_0+1}} &> l-\epsilon \\ \frac{U_{n_0+1}}{U_{n_0+2}} &> l-\epsilon \\ &\vdots \\ \frac{U_{n-1}}{U_n} &> l-\epsilon \end{aligned}\right\} \quad \forall\ n \geq n_0$$

Multiplying above, we get.

$$\frac{U_{n_0}}{U_n} > (l-\epsilon)^{n-n_0}$$

i.e $$\frac{U_{n_0}}{U_n} > \frac{(l-\epsilon)^n}{(l-\epsilon)^{n_0}}$$

i.e $$U_n < U_0 (l-\epsilon)^{n_0} \left(\frac{1}{l-\epsilon}\right)^n \quad \forall\ n \geq n_0$$

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Now, $\sum \left(\frac{1}{l-\epsilon}\right)^n$ is Geom. series with $r = \frac{1}{l-\epsilon} < 1$

& hence converges

$\therefore$ By Basic comp. test

$\sum u_n$ also converges

Case (i)(b) $l = \infty$

Choose any $k > 1$

As $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \infty$ $\therefore$ corresponding to above $k > 0$,

$\exists\ n_0 \in \mathbb{N}$ s.t. $\frac{u_n}{u_{n+1}} > k \quad \forall\ n \geq n_0$

Proceeding as in case (i)(a) (with k in place of $l-\epsilon$), we get

$$u_n < u_{n_0} k^{n_0} \left(1/k\right)^n \quad \forall\ n \geq n_0$$

Now $\sum (1/k)^n$ is a geometric series with $r = 1/k < 1$

& Hence converges.

$\therefore$ By Basic comp-test, $\sum u_n$ also converges

Case (ii) $l < 1$

In this case, we choose $\epsilon > 0$ s.t. $l + \epsilon < 1$

Now proceeding as in case (i)(a), we get req. conclusion.

Case (iii) $l = 1$

Consider the series $\sum 1/n^2$ (Convergent as $p = 2 > 1$)

Here $u_n = 1/n^2$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = \frac{n^2(1+1/n)^2}{n^2} = 1$$

Next consider the series $\sum 1/n$ (divergent as $p = 1$)

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Here too $l = 1$
Thus, series for which $l = 1$ may converges or diverges.

ALTERNATING SERIES

Def^n An alternating series is of the form
$u_1 - u_2 + u_3 - u_4 + \cdots$ $\quad (u_n > 0 \;\; \forall\, n \in \mathbb{N})$

i.e $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} u_n$.

LEIBNITZ' TEST

let $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ be an alternating series (ie $u_n > 0$, $\forall\, n \in \mathbb{N}$)
suppose that
(i) $\langle u_n \rangle$ is $\downarrow$ $\quad$ (i.e $u_1 \geq u_2 \geq u_3 \geq \cdots$)
(ii) $\displaystyle\lim_{n \to \infty} u_n = 0$

Then $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ converges.

Eg: $1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \cdots$

given series is $\sum (-1)^{n-1} u_n$, where $u_n = \frac{1}{4n-3}$

(i) $u_{n+1} = \frac{1}{4(n+1)-3} = \frac{1}{4n+1}$

$4n - 3 < 4n+1 \Rightarrow \frac{1}{4n-3} > \frac{1}{4n+1}$ i.e $u_n > u_{n+1}$

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$\therefore \langle u_n \rangle$ is $\downarrow$

(ii) $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \dfrac{1}{4n-3} = 0$

$\therefore$ By leibnitz Test, $\Sigma (-1)^{n-1} u_n$ converges

Eg: $\dfrac{1}{1\cdot 2} - \dfrac{1}{3\cdot 4} + \dfrac{1}{5\cdot 6} - \cdots$

given series is $\sum (-1)^{n-1} u_n$

where $u_n = \dfrac{1}{(2n-1)2n}$

(i) $u_{n+1} = \dfrac{1}{(2n+1)(2n+2)}$

$$\left.\begin{array}{l} 2n-1 < 2n+1 \\ 2n < 2n+2 \end{array}\right\} \Rightarrow (2n-1)(2n) < (2n+1)(2n+2)$$

$$\Rightarrow \frac{1}{(2n-1)2n} > \frac{1}{(2n+1)(2n+2)}$$

i.e $u_n > u_{n+1}$

$\therefore \langle u_n \rangle$ is $\downarrow$

(ii) $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \dfrac{1}{(2n-1)2n} = 0$

$\therefore$ By leibnitz Test

$\sum\limits_{n=1}^{\infty} (-1)^{n-1} u_n$ converges.

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ARBITRARY SERIES

Cauchy's Gen Principle of convergence

$\sum u_n$ converges iff for each $\epsilon > 0$, $\exists$ $n_0 \in \mathbb{N}$ s.t.

$$|u_{n_0+1} + u_{n_0+2} + \cdots + u_n| < \epsilon \quad \forall\ n \geq n_0$$

Proof :

$\sum u_n$ converges

$\Leftrightarrow$ its sequence of partial sum $\langle S_n \rangle$ converges

$\Leftrightarrow$ $\langle S_n \rangle$ is cauchy (By cauchy's conv. criterion for seq^n.)

$\Leftrightarrow$ for each $\epsilon > 0$, $\exists$ $n_0 \in \mathbb{N}$ s.t $|S_n - S_{n_0}| < \epsilon$ $\forall$ $n \geq n_0$

$\Leftrightarrow$ for each $\epsilon > 0$, $\exists$ $n_0 \in \mathbb{N}$ s.t.

$$|u_{n_0+1} + u_{n_0+2} + \cdots + u_n| < \epsilon \quad \forall\ n \geq n_0.$$

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Absolutely convergent

Def^n: Σu_n is said to be absolutely convergent iff $\Sigma |u_n|$ converges.

(Not absolutely converges iff $\Sigma |u_n|$ diverges)

Thm: Absolute convergence $\Rightarrow$ Convergence.

Proof

Given: Σu_n is ~~abs~~ absolute convergent

i.e $|\Sigma u_n|$ converges.

$\therefore$ By cauchy's Gen. Principle of conv.

for each $\epsilon > 0$, $\exists$ $n_0 \in \mathbb{N}$ s.t.

$$\big| |u_{n_0+1}| + |u_{n_0+2}| + \cdots + |u_n| \big| < \epsilon \quad \forall n \geq n_0$$

i.e $|u_{n_0+1}| + |u_{n_0+2}| + \cdots + |u_n| < \epsilon \quad \forall n \geq n_0 \quad$ — (1)

consider, $|u_{n_0+1} + \cdots + u_n|$

$\leq |u_{n_0+1}| + |u_{n_0+2}| + \cdots + |u_n|$

$< \epsilon \quad \forall n \geq n_0 \qquad$ by (1)

$\therefore$ By cauchy's Gen prin. of conv.

Σu_n converges.

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Note: Converse is not true.

example of convergent but not abs. convergt.

Consider, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$

Convergent: the above series is $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$, where $u_n = \frac{1}{n}$

(i) clearly $u_{n+1} < u_n$ ie $\langle u_n \rangle \downarrow$

(ii) $\lim_{n \to \infty} u_n = 0$

$\therefore$ By leibnitz test, $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ converges

Not abs. convergt.

Consider, $\sum_{n=1}^{\infty} |(-1)^{n-1} \frac{1}{n}| = \sum_{n=1}^{\infty} \frac{1}{n}$.

This diverges as $p = 1$

$\therefore \sum_{n=1}^{\infty} (-1)^{n-1} u_n$ is not abs. convgt.

CONDITIONALLY CONVERGENT

Def^n: A series $\sum u_n$ is said to be conditionally convgt. iff it is convergent but not absolutely convgt.

Eg: $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$

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Eg: Show that the series $\sum_{n=1}^{\infty} (-1)^{n-1} 1/n^p$ is absolutely convergent if $p>1$ & conditionally convgt if $p\le 1$.

Solⁿ Test for absolute convergent.

consider $\sum_{n=1}^{\infty} \left| (-1)^{n-1} 1/n^p \right| = \sum_{n=1}^{\infty} 1/n^p$

which converges if $p>1$
& diverges if $p\le 1$

$\therefore \sum_{n=1}^{\infty} (-1)^{n-1} 1/n^p$ is abs convgt if $p>1$
& not abs con. if $p\le 1$

Test for convergence

Consider $\sum_{n=1}^{\infty} (-1)^{n-1} 1/n^p = \sum_{n=1}^{\infty} (-1)^{n-1} U_n$ where $U_n = 1/n^p$

(i) clearly $\langle U_n \rangle \downarrow$
(ii) clearly $\lim_{n\to\infty} U_n = 0$

$\therefore$ By leibnitz test, $\sum_{n=1}^{\infty} (-1)^{n-1} U_n$ converges

Thus $p>1 \Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^p}$ is abs. convgt.

& $p\le 1 \Rightarrow$ " , conditionally convgt.