Infinite series

Def^n: An infinite series is an infinite sum

$$\sum_{n=1}^{\infty} U_n = U_1 + U_2 + \cdots + U_n + \cdots$$

Def^n: Let $\sum U_n$ be an infinite series
Define its sequence of Partial sums $\langle S_n \rangle$ as

$$S_1 = U_1$$
$$S_2 = U_1 + U_2$$
$$\vdots$$
$$S_n = U_1 + U_2 + \cdots + U_n$$

* An infinite series $\sum U_n$ is said to converges/ diverge/ oscillate according as its sequence of Partial sum $\langle S_n \rangle$ converges/ diverges/ oscillates resp.
Also sum of series = limit of its sequence of partial sum

Thm: Necessary Condition for convergence

A necessary condition for convergence of $\sum U_n$ is $\lim_{n \to \infty} U_n = 0$.

Proof: let $\sum U_n$ converge
i.e its sequence of partial sum $\langle S_n \rangle$ converges to l (say)
where

$$S_1 = U_1$$
$$S_2 = U_1 + U_2$$
$$\vdots$$
$$S_{n-1} = U_1 + U_2 + \cdots + U_{n-1}$$
$$S_n = U_1 + U_2 + \cdots + U_{n-1} + U_n$$

Page no. ____
Date ____

Clearly, $U_n = S_n - S_{n-1}$

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$= l - l$$

$$= 0$$

Eg. Show that the following series do not converge.

(i) $\sum \frac{n}{n+1}$ (ii) $\sum n^{1/n}$

(iii) $\sum \sin n\pi/3$ (iv) $\sum (-1)^n$

Solⁿ (i) $\lim_{n\to\infty} \frac{n}{n+1} \Rightarrow 1 \neq 0$

$\therefore$ $\sum U_n$ does not converge.

SERIES OF NON-NEGATIVE TERMS

let $\sum U_n$ be a series of non-negative terms,
i.e $U_n \geq 0 \ \forall n \in \mathbb{N}$

Then clearly its Sequence of partial sum $\langle S_n \rangle$ is increases. Hence

1.) $\langle S_n \rangle$ Cannot osillate, i.e it either converges or diverges to ∞

2.) A necessary and sufficient condition for $\langle S_n \rangle$ to converge is that it is bounded above.

Thm

* BASIC COMPARISON TEST

let ΣU_n and ΣV_n be 2 series of non negative terms suppose that $\exists K>0$ and $n_0 \in N$ s.t.

$$U_n \leq K V_n \quad \forall \; n \geq n_0 .$$

If ΣV_n converges, than ΣU_n also converges

Particular case of basic comparison test

Taking $n_0 = 1$

→ let ΣU_n and ΣV_n be two series of non. negative terms s.t.

$$U_n \leq K V_n \quad \forall \; n \in \mathbb{N} \quad (K>0)$$

(i) If ΣV_n converges, then ΣU_n also converges

(ii) if ΣU_n diverges then ΣV_n also diverges)

(only statement)

Thm: The "P- series"

$$\Sigma 1/n^P = 1 + 1/2^P + 1/3^P + \cdots + 1/n^P + \cdots$$

* The series $\Sigma 1/n^P$ Converges if $P>1$ and diverges if $P \leq 1$

Proof: case ① $P>1$

Page No. 5

Date

let $\langle S_n \rangle$ be the sequence of partial sum of $\sum 1/n^p$

i.e $S_n = 1 + 1/2^p + 1/3^p + \cdots + 1/n^p$

choose $k \in \mathbb{N}$ s.t. $n \leq 2^k - 1$

then $S_n \leq 1 + \left(\frac{1}{2^p} + 1/3^p\right) + \left(\frac{1}{4^p} + \cdots + 1/7^p\right) + \left(\frac{1}{8^p} + \cdots + 1/15^p\right)$

$+ \cdots + \left(\frac{1}{(2^{k-1})^p} + \cdots + \frac{1}{(2^k - 1)^p}\right)$

$$\leq 1 + \underbrace{\left(\frac{1}{2^p} + 1/2^p\right)}_{\text{2 Terms}} + \underbrace{\left(\frac{1}{4^p} + \cdots + 1/4^p\right)}_{\text{4 terms}} + \underbrace{\left(\frac{1}{8^p} + \cdots + 1/8^p\right)}_{\text{8 terms}}$$

$+ \cdots + \left(\frac{1}{(2^{k-1})^p} + \frac{1}{(2^{k-1})^p} + \cdots + 1/(2^{k-1})^p\right)$

$= 1 + 2/2^p + 4/4^p + \cdots + \frac{2^{k-1}}{(2^{k-1})^p}$

$= 1 + 2/2^p + (2/2^p)^2 + (2/2^p)^3 + \cdots + (2/2^p)^{k-1}$

$$= \frac{1\left(1 - (2/2^p)^k\right)}{1 - (2/2^p)}$$

$$= \frac{1}{1 - 2/2^p} - \frac{(2/2^p)^k}{1 - (2/2^p)} \qquad \left[1 - (2/2^p) > 0\right]$$

$$< \frac{1}{1 - 2/2^p}$$

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$\therefore \langle S_n \rangle$ is bounded above.

Also, $\langle S_n \rangle$ is increase sequence. $[\because 1/n^p > 0 \ \forall n \in \mathbb{N}]$

$\therefore \langle S_n \rangle$ Converges

i.e $\sum 1/n^p$ Converges.

Case (ii) $p = 1$

The series $\sum 1/n^p$ becomes $\sum 1/n = 1 + 1/2 + \cdots + 1/n + \cdots$

which diverges

case (iii) $p < 1$

The series is $\sum 1/n^p = 1 + 1/2^p + 1/3^p + \cdots \ (p<1)$ — ①

Consider the series $\sum 1/n = 1 + 1/2 + \cdots$ — ②

Clearly, ~~count~~ each term of ① is $\geq$ corresponding term of ②

Also ② diverges (by case (ii))

By basic comparison test

① also diverges

Date

Thm: The Geometric series $1+x+x^2+x^3+\cdots$ to ∞

(i) Converges if $-1<x<1$ i.e $|x|<1$

(ii) diverges if $x\geq 1$

(iii) oscillates finitely if $x=-1$

(iv) oscillates infinitely if $x<-1$

Proof: When $|x|<1$

$|x|<1$, $x^n \to 0$ as $n\to\infty$

$S_n = 1+x+x^2+\cdots$ to n terms

$$= \frac{1(1-x^n)}{1-x} = \frac{1}{1-x} - \frac{x^n}{1-x}.$$

$$\lim_{n\to\infty} S_n = \frac{1}{1-x}.$$

$\Rightarrow$ the seq^n $\{S_n\}$ is convergent.

$\Rightarrow$ the given series is convergent.

(ii) When $x\geq 1$.

sub case I when $x=1$.

$$S_n = \underbrace{1+1+\cdots+1}_{n \text{ terms}}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} n = \infty$$

$\Rightarrow$ the given series diverges to ∞

sub case II when $x>1$, $x^n\to\infty$ as $n\to\infty$

$S_n = 1+x+\cdots+x^n$

$$= \frac{1(x^n-1)}{x-1}$$

$\lim_{n\to\infty} S_n = \infty$ $\Rightarrow$ the given series diverges to ∞.

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(iii) When $x = -1$

$$S_n = 1 - 1 + 1 - 1 + \cdots \text{ to } n \text{ terms}$$

$$= \begin{cases} 1 & \text{, if } n \text{ is odd} \\ 0 & \text{, if } n \text{ is even.} \end{cases}$$

$S_{2n-1} \to 1$ and $S_{2n} \to 0$

$\Rightarrow$ the seq^n $\{S_n\}$ oscillates finitely

$\Rightarrow$ the given series " "

(iv) When $x < -1$

$\Rightarrow -x > 1$

let $r = -x$ then $r > 1$

$r^n \to \infty$ as $n \to \infty$

$$S_n = 1 + x + x^2 + \cdots + x^n$$

$$= \frac{1 - x^n}{1 - x} = \frac{1 - (-r)^n}{1 + r}$$

$$= \frac{1 - r^n}{1 + r} \text{ or } \frac{1 + r^n}{1 + r}$$ according as n is even or n is odd.

$S_{2n} \to -\infty$ and $S_{2n-1} \to \infty$

$\Rightarrow$ the seq^n $\{S_n\}$ oscillates infinitely

$\Rightarrow$ the given series " " .

Date

Thm : LIMIT COMPARSION TEST

Statement :

let Σu_n and Σv_n be 2 series of +ve terms

If $\lim\limits_{n\to\infty} \frac{u_n}{v_n}$ is finite and non zero , then

Σu_n and Σv_n either both converge or both diverges (i.e have same behaviour)

Proof:

let $l = \text{limit}_{n\to\infty} \frac{u_n}{v_n}$, then l is finite and nonzero (given)

Now, $\left.\begin{array}{l} u_n > 0 \\ v_n > 0 \end{array}\right\} \Rightarrow \frac{u_n}{v_n} > 0 \quad \forall\, n \in \mathbb{N}$

$\Rightarrow \lim\limits_{n\to\infty} \frac{u_n}{v_n} \geq 0$, i.e $l \geq 0$.

Also . l is non zero

$\Rightarrow l > 0$

Now, choose $\epsilon > 0$ s.t $l - \epsilon > 0$

As $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = l$

$\therefore$ corresp. to above $\epsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ s.t.

$$l - \epsilon < \frac{u_n}{v_n} < l + \epsilon \quad \forall\, n \geq n_0$$

This gives the foll.

(i) $\quad u_n < (l+\epsilon)\, v_n \quad\quad \forall\, n \geq n_0$

Page No.
Date

$\therefore$ By Basic comp. Test,

$$\left.\begin{array}{l}\sum v_n \text{ converges} \Rightarrow \sum u_n \text{ converges} \\ \& \sum u_n \text{ diverges} \Rightarrow \sum v_n \text{ diverges}\end{array}\right\} \text{ — Ⓐ}$$

2.) $(l-\epsilon) v_n < u_n \quad \forall \; n \geq n_0$

i.e $v_n < \left(\frac{1}{l-\epsilon}\right) u_n \quad \forall \; n \geq n_0 \quad (\because l-\epsilon > 0)$

By basic comp. test.

$$\left.\begin{array}{l}\sum u_n \text{ converges} \Rightarrow \sum v_n \text{ converges} \\ \& \sum v_n \text{ diverges} \Rightarrow \sum u_n \text{ diverges}\end{array}\right\} \text{ — Ⓑ}$$

from Ⓐ & Ⓑ together give required result.

Eg: 1.) Test for convergence

$$\frac{1}{5} + \frac{\sqrt{2}}{7} + \frac{\sqrt{3}}{9} + \cdots$$

Soln

$$u_n = \frac{\sqrt{n}}{2n+3} \Rightarrow \frac{\sqrt{n}}{n(2+3/n)} = \frac{1}{n^{1/2}(2+3/n)}$$

Take, $v_n = 1/n^{1/2}$

$$\frac{u_n}{v_n} = \frac{1}{2+3/n}$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = 1/2 \text{ (which is finite \& nonzero)}$$

$\therefore$ By comparison Test,

$\sum u_n$ and $\sum v_n$ have same behaviour

Now $\sum v_n$ diverges (as $p = 1/2 < 1$, By p- Test)

$\Rightarrow \sum u_n$ also converges.

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Eg 2. Test for convergence

$$\frac{1}{3.7} + \frac{1}{4.9} + \frac{1}{5.11} + \cdots$$

Sol^n

$$U_n = \frac{1}{(2+n)(2n+5)} = \frac{1}{n^2\left(1+2/n\right)\left(2+5/n\right)}$$

Take $V_n = \frac{1}{n^2}$

$$\frac{U_n}{V_n} = \frac{1}{(1+2/n)(2+5/n)}$$

$\therefore \lim\limits_{n\to\infty} \frac{U_n}{V_n} = 1/2$ (finite and non zero)

By Comparison test

ΣU_n and ΣV_n have same behaviour.

Now ΣV_n converges (as $p=2>1$)

$\Rightarrow \Sigma U_n$ also converges.

Eg 3

Test for convergence : $\Sigma\left(\sqrt{n^3+1} - \sqrt{n^3}\right)$

Sol^n:

$$U_n = \sqrt{n^3+1} - \sqrt{n^3}$$

$$= \left(\sqrt{n^3+1} - \sqrt{n^3}\right)\left(\frac{\sqrt{n^3+1}+\sqrt{n^3}}{\sqrt{n^3+1}+\sqrt{n^3}}\right)$$

$$= \frac{(n^3+1) - n^3}{\sqrt{n^3+1} + \sqrt{n^3}}$$

$$= \frac{1}{\sqrt{n^3+1} + \sqrt{n^3}}$$

$$\Rightarrow \frac{1}{n^{3/2}\left(\sqrt{1+1/n^3} + 1\right)}$$

Take $V_n = 1/n^{3/2}$

Page No.
Date

$$\frac{u_n}{v_n} = \frac{1}{\sqrt{1+1/n^3}+1}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = 1/2 \text{, finite and non zero.}$$

$\therefore$ By Comparison test
Σu_n and Σv_n have same behaviour.
Now Σv_n converges (as $p = 3/2 > 1$)
$\therefore$ Σu_n also converges.