

VECTOR-VALUED FUNCTIONS

In this topic we will consider functions whose values are vectors.

Definition 1: A vector-valued function in 3-space is a function of the form

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

where the functions $x(t)$, $y(t)$ and $z(t)$ are real-valued functions of the parameter t . The functions $x(t)$, $y(t)$ and $z(t)$ are called the *component functions* or the *components* of $\mathbf{r}(t)$.

Vector-valued functions are often denoted as $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$.

The *domain* of a vector-valued function $\mathbf{r}(t)$ is the set of all t 's for which all the components functions are defined. If $\mathbf{r}(t)$ is defined in terms of component functions and the domain is not specified explicitly, then it will be understood that the domain is the intersection of the natural domains of the component functions; this is called the *natural domain* of $\mathbf{r}(t)$.

Example 1: Find the natural domain of

$$\mathbf{r}(t) = \langle \ln |t - 1|, e^t, \sqrt{t} \rangle$$

Solution. The natural domains of the component functions

$$x(t) = \ln |t - 1|, y(t) = e^t, z(t) = \sqrt{t}$$

are

$$(-\infty, 1) \cup (1, +\infty), (-\infty, +\infty), [0, +\infty)$$

respectively. The intersection of these sets is

$$[0,1) \cup (1, +\infty)$$

so the natural domain of $\mathbf{r}(t)$ consists of all values of t such that

$$0 \leq t < 1 \text{ or } t > 1.$$

GRAPHS OF VECTOR-VALUED FUNCTIONS

If $\mathbf{r}(t)$ is a vector-valued function in 2-space or 3-space, then we define the **graph** of $\mathbf{r}(t)$ to be the parametric curve described by the component functions for $\mathbf{r}(t)$. For example, if

$$\mathbf{r}(t) = \langle 1 - t, 3t, 2t \rangle$$

then the graph of $\mathbf{r} = \mathbf{r}(t)$ is the graph of the parametric equations

$$x = 1 - t, y = 3t, z = 2t$$

Thus, the graph of $\mathbf{r}(t)$ is the line in figure 1 below.

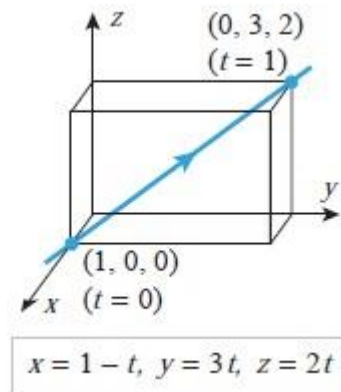


Figure 1

If the curve C in 3-space is the graph of

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

and if we position $\mathbf{r}(t)$ so its initial point is at the origin, then its terminal point will fall on the curve C . Thus, when $\mathbf{r}(t)$ is positioned with its initial point at the origin, its terminal point will trace out the curve C as the parameter t varies, in which case we call $\mathbf{r}(t)$ the *radius vector* or the *position vector* for C .

Example 2: Sketch the graph and a radius vector of

a) $\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j}, 0 \leq t \leq 2\pi$

b) $\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + 2\mathbf{k}, 0 \leq t \leq 2\pi$

Solution. (a) The corresponding parametric equations are

$$x = \cos t, y = \sin t \quad (0 \leq t \leq 2\pi)$$

so the graph is a circle of radius 1, centered at the origin, and oriented counterclockwise. The graph and a radius vector are shown below in figure 2.

(b) The corresponding parametric equations are

$$x = \cos t, y = \sin t, z = 2 \quad (0 \leq t \leq 2\pi)$$

From the third equation, the tip of the radius vector traces a curve in the plane $z = 2$, and from the first two equations, the curve is a circle of radius 1 centered at the point $(0, 0, 2)$ and traced counterclockwise looking down the z -axis. The graph and a radius vector are shown in the figure 3.

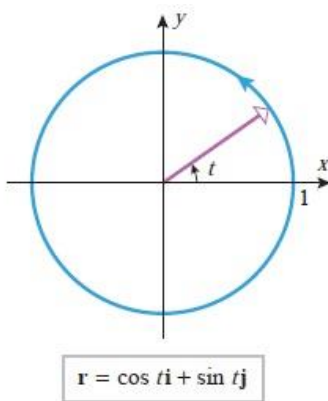


Figure 2

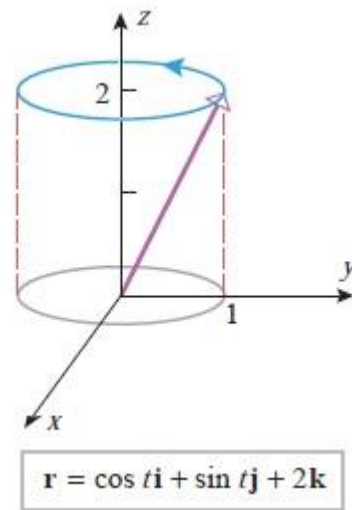


Figure 3

VECTOR FORM OF A LINE SEGMENT

We recall that if \mathbf{r}_0 is a vector in 2-space or 3-space with its initial point at the origin, then the line that passes through the terminal point of \mathbf{r}_0 and is parallel to the vector \mathbf{v} can be expressed in vector form as

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

In particular, if \mathbf{r}_0 and \mathbf{r}_1 are vectors in 2-space or 3-space with their initial points at the origin, then the line that passes through the terminal points of these vectors can be expressed in vector form as

$$\mathbf{r} = \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0)$$

or

$$\mathbf{r} = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \quad (1)$$

as shown in figure 4. Equation (1) represents *two-point vector form of a line* passing through the points \mathbf{r}_0 and \mathbf{r}_1 .

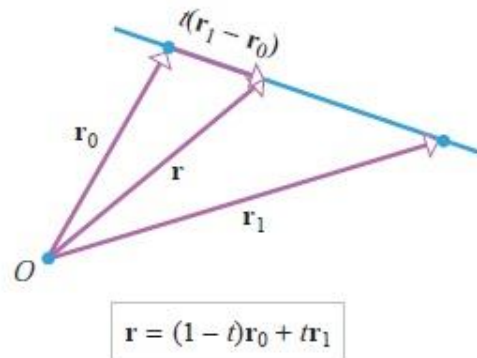


Figure 4

LIMIT AND CONTINUITY

We now aim to develop a notion of what it means for a vector-valued function $\mathbf{r}(t)$ in 2-space or 3-space to approach a limiting vector \mathbf{L} as t approaches a number a . That is, we want to define

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \mathbf{L}$$

If $\mathbf{r}(t)$ and \mathbf{L} are positioned so that their initial points are at the origin and this limit can be interpreted to mean that the terminal point of $\mathbf{r}(t)$ approaches the terminal point of \mathbf{L} as t approaches a or, equivalently, that the vector $\mathbf{r}(t)$ approaches the vector \mathbf{L} in both length and direction at t approaches a (See figure 5). Algebraically, this is equivalent to stating that

$$\lim_{t \rightarrow a} \|\mathbf{r}(t) - \mathbf{L}\| = 0 \quad (\text{See figure 6})$$

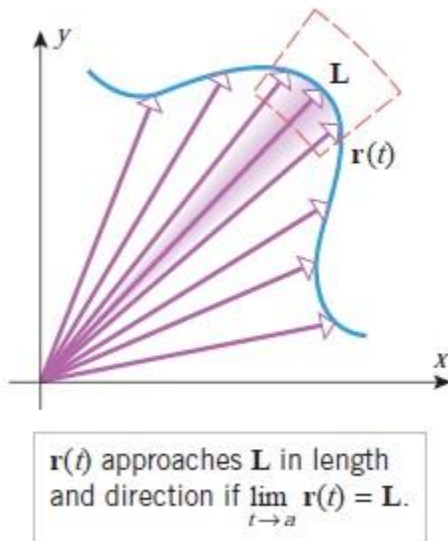


Figure 5

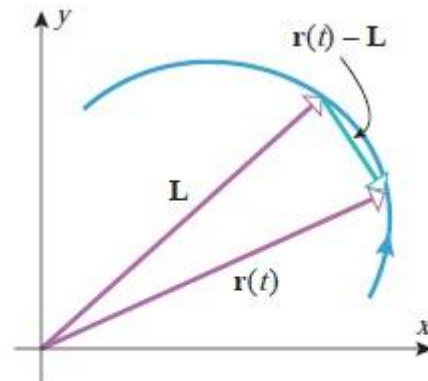


Figure 6

Definition 2: Let $\mathbf{r}(t)$ be a vector-valued function that is defined for all t in some open interval containing the number a , except that $\mathbf{r}(t)$ need not be defined at a .

We will write

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \mathbf{L}$$

if and only if

$$\lim_{t \rightarrow a} \|\mathbf{r}(t) - \mathbf{L}\| = 0 .$$

It is clear intuitively that $\mathbf{r}(t)$ will approach a limiting vector \mathbf{L} as t approaches a if and only if the component functions of $\mathbf{r}(t)$ approach the corresponding components of \mathbf{L} . Next theorem states this logic.

Theorem 1 (Only Statement):

If

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},$$

then

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \lim_{t \rightarrow a} x(t)\mathbf{i} + \lim_{t \rightarrow a} y(t)\mathbf{j} + \lim_{t \rightarrow a} z(t)\mathbf{k}$$

provided the limits of the component functions exist. Conversely, the limits of the component functions exist provided $\mathbf{r}(t)$ approaches a limiting vector as t approaches a .

Example 3: If $\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k}$, find $\lim_{t \rightarrow a} \mathbf{r}(t)$ where $a = \pi/4$.

Solution. By Theorem 1,

$$\lim_{t \rightarrow \pi/4} \mathbf{r}(t) = \left(\lim_{t \rightarrow \pi/4} \cos t \right) \mathbf{i} + \left(\lim_{t \rightarrow \pi/4} \sin t \right) \mathbf{j} + \left(\lim_{t \rightarrow \pi/4} t \right) \mathbf{k}$$

$$= \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \frac{\pi}{4}\mathbf{k}$$

Example 4: Evaluate $\lim_{t \rightarrow 0} \mathbf{r}(t) = t^2\mathbf{i} + e^t\mathbf{j} - (2 \cos \pi t)\mathbf{k}$.

Solution. By Theorem 1,

$$\begin{aligned} \lim_{t \rightarrow 0} \mathbf{r}(t) &= (\lim_{t \rightarrow 0} t^2)\mathbf{i} + (\lim_{t \rightarrow 0} e^t)\mathbf{j} - (\lim_{t \rightarrow 0} 2 \cos \pi t)\mathbf{k} \\ &= \mathbf{j} - 2\mathbf{k} \end{aligned}$$

Motivated by the definition of continuity for real-valued functions, we define a vector-valued function $\mathbf{r}(t)$ to be *continuous* at $t = a$ if

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \mathbf{r}(a).$$

It follows from Theorem 1 that a vector-valued function $\mathbf{r}(t)$ is *continuous* at $t = a$ if and only if each of its component function is *continuous* at $t = a$.

Example 5: Discuss the continuity of the vector-valued function given by

$$\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t\mathbf{k}, \quad \text{at } t = 0.$$

Solution. We have

$$\begin{aligned} \lim_{t \rightarrow 0} \mathbf{r}(t) &= (\lim_{t \rightarrow 0} \cos t)\mathbf{i} + (\lim_{t \rightarrow 0} \sin t)\mathbf{j} + (\lim_{t \rightarrow 0} t)\mathbf{k} \\ &= 1\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} \\ &= \mathbf{i} \end{aligned}$$

and

$$\begin{aligned} \mathbf{r}(0) &= \cos 0 \mathbf{i} + \sin 0 \mathbf{j} + 0\mathbf{k} \\ &= 1\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} \\ &= \mathbf{i} \end{aligned}$$

Hence, $\lim_{t \rightarrow 0} \mathbf{r}(t) = \mathbf{r}(0)$. Thus, $\mathbf{r}(t)$ is *continuous* at $t = 0$.