

Sequences and Series of Functions -1

Consider sequences and series whose terms depend on a variable, i.e., those whose terms are real valued functions defined on an interval as domain. The sequences and series are denoted by $\{f_n\}$ and $\sum f_n$ respectively.

Point-wise Convergence

Definition. Let $\{f_n\}$, $n = 1, 2, 3, \dots$ be a sequence of functions, defined on an interval I , $a \leq x \leq b$. If there exists a real valued function f with domain I such that

$$f(x) = \lim_{n \rightarrow \infty} \{f_n(x)\}, \quad \forall x \in I$$

Then the function f is called the limit or the point-wise limit of the sequence $\{f_n\}$ on $[a, b]$, and the sequence $\{f_n\}$ is said to be point-wise convergent to f on $[a, b]$.

Similarly, if the series $\sum f_n$ converges for every point $x \in I$, and we define

$$f(x) = \sum_{n=0}^{\infty} f_n(x), \quad \forall x \in [a, b]$$

the function f is called the sum or the point-wise sum of the series $\sum f_n$ on $[a, b]$.

Definition. If a sequence of functions $\{f_n\}$ defined on $[a, b]$, converges pointwise to f , then to each $\epsilon > 0$ and to each $x \in [a, b]$, there corresponds an integer N such that

$$|f_n(x) - f(x)| < \epsilon, \quad \forall n \geq N \quad (1.1)$$

Remark:

1. The limit of differentials may not equal to the differential of the limit.

Consider the sequence $\{f_n\}$, where $f_n(x) = \frac{\sin nx}{\sqrt{n}}$, (x real).

It has the limit

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) = 0$$

$\therefore f'(x) = 0$, and so $f'(0) = 0$

But

$$f'_n(x) = \sqrt{n} \cos nx$$

so that

$$f'_n(0) = \sqrt{n} \rightarrow \infty \text{ as } n \rightarrow \infty$$

Thus at $x = 0$, the sequence $\{f'_n(x)\}$ diverges whereas the limit function $f'(x) = 0$,

2. Each term of the series may be continuous but the sum f may not.

Consider the series

$$\sum_{n=0}^{\infty} f_n, \text{ where } f_n(x) = \frac{x^2}{(1+x^2)^n} \text{ (x real)}$$

At $x = 0$, each $f_n(x) = 0$, so that the sum of the series $f(0) = 0$.

For $x \neq 0$, it forms a geometric series with common ratio $1/(1+x^2)$, so that its sum function $f(x) = 1+x^2$.

Hence,

$$f(x) = \begin{cases} 1+x^2, & x \neq 0 \\ 0 & , x = 0 \end{cases}$$

3. The limit of integrals is not equal to the integral of the limit.

Consider the sequence $\{f_n\}$, where

$$f_n(x) = nx(1-x^2)^n, 0 \leq x \leq 1, n = 1, 2, 3, \dots$$

For $0 < x \leq 1$, $\lim_{n \rightarrow \infty} f_n(x) = 0$

At $x = 0$, each $f_n(0) = 0$, so that $\lim_{n \rightarrow \infty} f_n(0) = 0$

Thus the limit function $f(x) = \lim_{n \rightarrow \infty} f_n(x) = 0$, for $0 \leq x \leq 1$

$$\therefore \int_0^1 f(x) dx = 0$$

Again,

$$\int_0^1 f_n(x) dx = \int_0^1 nx(1-x^2)^n dx = \frac{n}{2n+2}$$

so that

$$\lim_{n \rightarrow \infty} \left\{ \int_0^1 f_n(x) dx \right\} = \frac{1}{2}$$

Thus,

$$\lim_{n \rightarrow \infty} \left\{ \int_0^1 f_n dx \right\} \neq \int_0^1 f dx = \int_0^1 \left[\lim_{n \rightarrow \infty} \{f_n\} \right] dx$$

Uniform Convergence

Definition. A sequence of functions $\{f_n\}$ is said to converge uniformly on an interval $[a, b]$ to a function f if for any $\varepsilon > 0$ and for all $x \in [a, b]$ there exists an integer N (independent of x but dependent on ε) such that for all $x \in [a, b]$

$$|f_n(x) - f(x)| < \varepsilon, \quad \forall n \geq N \quad (1)$$

Remark. Every uniformly convergent sequence is pointwise convergent, and the uniform limit function is same as the pointwise limit function. But the converse is not true. However non-pointwise convergence implies non-uniform convergence .

Definition. A series of functions $\sum f_n$ is said to converge uniformly on $[a, b]$ if the sequence $\{S_n\}$ of its partial sums, defined by

$$S_n(x) = \sum_{i=1}^n f_i(x)$$

converges uniformly on $[a, b]$.

Definition. A series of functions $\sum f_n$ converges uniformly to f on $[a, b]$ if for $\varepsilon > 0$ and all $x \in [a, b]$ there exists an integer N (independent of x and dependent on ε) such that for all x in $[a, b]$

$$|f_1(x) + f_2(x) + \dots + f_n(x) - f(x)| < \varepsilon, \text{ for } n \geq N$$

Theorem (Cauchy's Criterion for Uniform Convergence). The sequence of functions $\{f_n\}$ defined on $[a, b]$ converges uniformly on $[a, b]$ if and only if for every $\varepsilon > 0$ and for all $x \in [a, b]$, there exists an integer N such that

$$|f_{n+p}(x) - f_n(x)| < \varepsilon, \quad \forall n \geq N, p \geq 1 \quad \dots(1)$$

Proof. Let the sequence $\{f_n\}$ uniformly converge on $[a, b]$ to the limit function f , so that for a given $\varepsilon > 0$, and for all $x \in [a, b]$, there exist integers n_1, n_2 such that

$$|f_n(x) - f(x)| < \varepsilon/2, \quad \forall n \geq n_1$$

and

$$|f_{n+p}(x) - f(x)| < \varepsilon/2, \quad \forall n \geq n_2, p \geq 1$$

Let $N = \max(n_1, n_2)$.

$$\begin{aligned} \therefore |f_{n+p}(x) - f_n(x)| &\leq |f_{n+p}(x) - f(x)| + |f_n(x) - f(x)| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon, \quad \forall n \geq N, p \geq 1 \end{aligned}$$

Conversely. Let the given condition hold so by Cauchy's general principle of convergence, $\{f_n\}$ converges for each $x \in [a, b]$ to a limit, say f and so the sequence converges pointwise to f .

For a given $\varepsilon > 0$, let us choose an integer N such that (1) holds. Fix n , and let $p \rightarrow \infty$ in (1). Since $f_{n+p} \rightarrow f$ as $p \rightarrow \infty$, we get

$$|f(x) - f_n(x)| < \varepsilon \quad \forall n \geq N, \text{ all } x \in [a, b]$$

which proves that $f_n(x) \rightarrow f(x)$ uniformly on $[a, b]$.

Remark. Other form of this theorem is :

The sequence of functions $\{f_n\}$ defined on $[a, b]$ converges uniformly on $[a, b]$ if and only if for every $\varepsilon > 0$ and for all $x \in [a, b]$, there exists an integer N such that

$$|f_n(x) - f_m(x)| < \varepsilon, \quad \forall n, m \geq N$$

Theorem 2. A series of functions $\sum f_n$ defined on $[a, b]$ converges uniformly on $[a, b]$ if and only if for every $\varepsilon > 0$ and for all $x \in [a, b]$, there exists an integer N such that

$$|f_{n+1}(x) + f_{n+2}(x) + \dots + f_{n+p}(x)| < \varepsilon, \quad \forall n \geq N, p \geq 1 \quad \dots(2)$$

Proof. Taking the sequence $\{S_n\}$ of partial sums of functions $\sum f_n$, defined by

$$S_n(x) = \sum_{i=1}^n f_i(x)$$

And applying above theorem, we get the result.

Example . Show that the sequence $\{f_n\}$, where

$$f_n(x) = \frac{nx}{1+n^2x^2}, \text{ for } x \in [a, b].$$

is not uniformly convergent on any interval $[a, b]$ containing 0.

Solution. The sequence converges pointwise to f , where $f(x) = 0, \quad \forall \text{ real } x.$

Let $\{f_n\}$ converge uniformly in any interval $[a, b]$, so that the pointwise limit is also the uniform limit. Therefore for given $\varepsilon > 0$, there exists an integer N such that for all $x \in [a, b]$, we have

$$\left| \frac{nx}{1+n^2x^2} - 0 \right| < \varepsilon, \quad \forall n \geq N$$

If we take $\varepsilon = \frac{1}{3}$, and t an integer greater than N such that $1/t \in [a, b]$, we find on

taking $n = t$ and $x = 1/t$, that

$$\frac{nx}{1+n^2x^2} = \frac{1}{2} < \frac{1}{3} = \varepsilon.$$

which is a contradiction and so the sequence is not uniformly convergent in the interval $[a, b]$, having the point $1/t$. But since $1/t \rightarrow 0$, the interval $[a, b]$ contains 0. Hence the sequence is not uniformly convergent on any interval $[a, b]$ containing 0.

Example . The sequence $\{f_n\}$, where

$$f_n(x) = x^n$$

is uniformly convergent on $[0, k]$, $k < 1$ and only pointwise convergent on $[0, 1]$.

Solution.

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

Thus the sequence converges pointwise to a discontinuous function on $[0, 1]$

Let $\varepsilon > 0$ be given.

For $0 < x \leq k < 1$, we have

$$|f_n(x) - f(x)| = x^n < \varepsilon$$

if

$$\left(\frac{1}{x}\right)^n > \frac{1}{\varepsilon}$$

or if

$$n > \log(1/\varepsilon)/\log(1/x)$$

This number, $\log(1/\varepsilon)/\log(1/x)$ increases with x , its maximum value being $\log(1/\varepsilon)/\log(1/k)$ in $]0, k]$, $k > 0$.

Let N be an integer $\geq \log(1/\varepsilon)/\log(1/k)$.

$$\therefore |f_n(x) - f(x)| < \varepsilon, \quad \forall n \geq N, 0 < x < 1$$

Again at $x = 0$,

$$|f_n(x) - f(x)| = 0 < \varepsilon, \quad \forall n \geq 1$$

Thus for any $\varepsilon > 0$, $\exists N$ such that for all $x \in [0, k]$, $k < 1$

$$|f_n(x) - f(x)| < \varepsilon, \quad \forall n \geq N$$

Therefore, the sequence $\{f_n\}$ is uniformly convergent in $[0, k]$, $k < 1$.

However, the number $\log(1/\varepsilon)/\log(1/x) \rightarrow \infty$ as $x \rightarrow 1$ so that it is not possible to find an integer N such that $|f_n(x) - f(x)| < \varepsilon$, for all $n \geq N$ and all x in $[0, 1]$. Hence the sequence is not uniformly convergent on any interval containing 1 and in particular on $[0, 1]$.

Example . Show that the sequence $\{f_n\}$, where

$$f_n(x) = \frac{1}{x+n}$$

is uniformly convergent in any interval $[0, b]$, $b > 0$.

Solution. The limit function is

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) = 0 \quad \forall x \in [0, b]$$

so that the sequence converges pointwise to 0.

For any $\varepsilon > 0$,

$$|f_n(x) - f(x)| = \frac{1}{x+n} < \varepsilon$$

if $n > (1/\varepsilon) - x$, which decreases with x , the maximum value being $1/\varepsilon$.

Let N be an integer $\geq 1/\varepsilon$, so that for $\varepsilon > 0$, there exists N such that

$$|f_n(x) - f(x)| < \varepsilon, \quad \forall n \geq N$$

Hence the sequence is uniformly convergent in any interval $[0, b]$, $b > 0$.

Example . The series $\sum f_n$, whose sum to n terms, $S_n(x) = nxe^{-nx^2}$, is pointwise and not uniformly convergent on any interval $[0, k]$, $k > 0$.

Solution. The pointwise sum $S(x) = \lim_{n \rightarrow \infty} S_n(x) = 0$, for all $x \geq 0$. Thus the series converges pointwise to 0 on $[0, k]$.

Let us suppose, if possible, the series converges uniformly on $[0, k]$, so that for any $\varepsilon > 0$, there exists an integer N such that for all $x \geq 0$,

$$|S_n(x) - S(x)| = nxe^{-nx^2} < \varepsilon, \quad \forall n \geq N \quad \dots(*)$$

Let N_0 be an integer greater than N and $e^2\varepsilon^2$, then for $x = 1/\sqrt{N_0}$ and $n = N_0$, (*) gives

$$\sqrt{N_0}/e < \varepsilon \Rightarrow N_0 < e^2\varepsilon^2$$

so we arrive at a contradiction. Hence the series is not uniformly convergent on $[0, k]$.

Note . The interval of uniform convergence is always to be a closed interval, that is it must include the end points. But the interval for pointwise or absolute convergence can be of any type.

Theorem 3. Let $\{f_n\}$ be a sequence of functions, such that

$$\lim_{n \rightarrow \infty} f_n(x) = f(x), \quad x \in [a, b]$$

and let

$$M_n = \text{Sup}_{x \in [a, b]} |f_n(x) - f(x)|$$

Then $f_n \rightarrow f$ uniformly on $[a, b]$ if and only if $M_n \rightarrow 0$ as $n \rightarrow \infty$.

Proof. Let $f_n \rightarrow f$ uniformly on $[a, b]$, so that for a given $\varepsilon > 0$, there exists an integer N such that

$$\begin{aligned} & |f_n(x) - f(x)| < \varepsilon, \quad \forall n \geq N, \quad \forall x \in [a, b] \\ \Rightarrow & M_n = \sup_{x \in [a, b]} |f_n(x) - f(x)| \leq \varepsilon, \quad \forall n \geq N \\ \Rightarrow & M_n \rightarrow 0, \text{ as } n \rightarrow \infty \end{aligned}$$

Conversely. Let $M_n \rightarrow 0$, as $n \rightarrow \infty$, so that for any $\varepsilon > 0$, \exists an integer N such that

$$\begin{aligned} & M_n < \varepsilon, \quad \forall n \geq N \\ \Rightarrow & \sup_{x \in [a, b]} |f_n(x) - f(x)| < \varepsilon, \quad \forall n \geq N \\ \Rightarrow & |f_n(x) - f(x)| < \varepsilon, \quad \forall n \geq N, \quad \forall x \in [a, b] \\ \Rightarrow & f_n \rightarrow f \text{ uniformly on } [a, b]. \end{aligned}$$

Example. Show that 0 is a point of non-uniform convergence of the sequence $\{f_n\}$, where $f_n(x) = 1 - (1 - x^2)^n$.

Solution. We have

$$\begin{aligned} M_n &= \sup \{ |f_n(x) - f(x)| : x \in]0, \sqrt{2}[\} \\ &= \sup \{ (1 - x^2)^n : x \in]0, \sqrt{2}[\} \\ &\geq \left(1 - \frac{1}{n} \right)^n \quad \left[\text{Taking } x = \frac{1}{\sqrt{n}} \in]0, \sqrt{2}[\right] \\ &\rightarrow \frac{1}{e} \text{ as } n \rightarrow \infty. \end{aligned}$$

Thus M_n cannot tend to zero as $n \rightarrow \infty$.

It follows that the sequence is non-uniformly convergent.

Also as $n \rightarrow \infty$, $x \rightarrow 0$ and consequently 0 is a point of non-uniform convergence.

Example . Prove that the sequence $\{f_n\}$, where

$$f_n(x) = \frac{x}{1 + nx^2}, \quad x \text{ real}$$

converges uniformly on any closed interval I .

Here pointwise limit,

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) = 0, \quad \forall x$$

$$\begin{aligned} M_n &= \sup_{x \in I} |f_n(x) - f(x)| = \sup_{x \in I} \left| \frac{x}{1 + nx^2} \right| \\ &= \frac{1}{2\sqrt{n}} \rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

Hence $\{f_n\}$ converges uniformly on I .

$$\left[\frac{x}{1 + nx^2} \text{ attains the maximum value } \frac{1}{2\sqrt{n}} \text{ at } x = \frac{1}{\sqrt{n}}, \text{ i.e. at the origin.} \right]$$

Example . Show that the sequence $\{f_n\}$, where

$$f_n(x) = nxe^{-nx^2}, \quad x \geq 0$$

is not uniformly convergent on $[0, k]$, $k > 0$

Solution. $f(x) = \lim_{n \rightarrow \infty} f_n(x) = 0, \quad \forall x \geq 0$

Also nxe^{-nx^2} attains maximum value $\sqrt{\frac{n}{2e}}$ at $x = \frac{1}{\sqrt{2n}}$

Now

$$\begin{aligned} M_n &= \sup_{x \in [0, k]} |f_n(x) - f(x)| \\ &= \sup_{x \in [0, k]} nxe^{-nx^2} = \sqrt{\frac{n}{2e}} \rightarrow \infty \text{ as } n \rightarrow \infty \end{aligned}$$

Therefore the sequence is not uniformly convergent on $[0, k]$.

Example. Prove that the sequence $\{f_n\}$, where $f_n(x) = x^{n-1} (1 - x)$ converges uniformly in the interval $[0, 1]$.

Solution. Here $f(x) = \lim_{n \rightarrow \infty} x^{n-1} (1 - x) = 0 \quad \forall x \in [0, 1]$.

Let $y = |f_n(x) - f(x)| = x^{n-1} (1 - x)$

Now y is maximum or minimum when

$$\begin{aligned} \frac{dy}{dx} &= (n-1)x^{n-2}(1-x) - x^{n-1} = 0 \\ x^{n-2} [(n-1)(1-x) - x] &= 0 \end{aligned}$$

or $x = 0$ or $\frac{n-1}{n}$.

As so $\frac{d^2y}{dx^2} = -ve$ when $x = \frac{n-1}{n}$

$$\therefore M_n = \max y = \left(1 + \frac{1}{n}\right)^{n-1} \left(1 - \frac{n-1}{n}\right) \rightarrow \frac{1}{e} \times 0 = 0 \text{ as } n \rightarrow \infty.$$

Hence the sequence is uniformly convergent on $[0, 1]$ by M_n -test.

Example. Show that 0 is a point of non-uniform convergence of the sequence $\{f_n\}$, where $f_n(x) = 1 - (1 - x^2)^n$.

Solution. Here

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) = \begin{cases} 0 & \text{when } x = 0 \\ 1 & \text{when } 0 < |x| < \sqrt{2} \end{cases}$$

Suppose, if possible, that the sequence is uniformly convergent in a neighborhood $]0, k[$ of 0 where k is a number such that $0 < k < \sqrt{2}$. There exists therefore a positive integer m such that

$$|f_m(x) - f(x)| < \frac{1}{2}, \text{ taking } \epsilon = \frac{1}{2},$$

i.e. if $(1 - x^2)^m < \frac{1}{2}$ for every $x \in]0, k[$.

Since $(1 - x^2)^m \rightarrow 1$ as $x \rightarrow 0$, we arrive at a contradiction. Hence 0 is point of non-uniform convergence of the sequence.

Example. Test for uniform convergence the series

$$\sum_{n=0}^{\infty} x e^{-nx} \text{ in the closed interval } [0, 1].$$

Solution. Here $f_n(x) = \sum_{n=1}^{n-1} x e^{-nx} = \frac{x(-1/e^{nx})}{1 - 1/e^x}$

$$= \frac{x e^x}{e^x - 1} \left(1 + \frac{1}{e^{nx}}\right)$$

$$\text{Now } f(x) = \lim_{n \rightarrow \infty} f_n(x) = \begin{cases} 0 & \text{where } x = 0 \\ \frac{xe^x}{e^x - 1} & \text{when } 0 < x \leq 1 \end{cases}$$

We consider $0 < x \leq 1$. We have

$$\begin{aligned} M_n &= \sup \{ |f_n(x) - f(x)| : x \in [0, 1] \} \\ &= \sup \left\{ \frac{xe^x}{(e^x - 1)e^{nx}} : x \in [0, 1] \right\} \\ &\geq \frac{1/n \cdot e^{1/n}}{(e^{1/n} - 1)e} \quad \left(\text{Taking } x = \frac{1}{n} \in [0, 1] \right) \end{aligned}$$

$$\begin{aligned} \text{Now } \lim_{n \rightarrow \infty} \frac{1/n \cdot e^{1/n}}{(e^{1/n} - 1)e} & \quad \left[\text{Form } \frac{0}{0} \right] \\ &= \lim_{n \rightarrow \infty} \frac{1/n e^{1/n} (-1/n^2) + (-1/n^2) e^{1/n}}{e \cdot e^{1/n} - (-1/n^2)} \\ &= \lim_{n \rightarrow \infty} \frac{(1/n + 1)}{e} = \frac{(0 + 1)}{e} = \frac{1}{e}. \end{aligned}$$

Thus M_n does not tend zero as $n \rightarrow \infty$.

Hence the sequence is non-uniformly convergent by M_n -test.

Here 0 is a point of non-uniform convergence.

Example . The sequence $\{f_n\}$, where

$$f_n(x) = \frac{nx}{1 + n^2 x^2}$$

is not uniformly convergent on any interval containing zero.

Solution. Here

$$\lim_{n \rightarrow \infty} f_n(x) = 0, \quad \forall x$$

Now $\frac{nx}{1 + n^2 x^2}$ attains the maximum value $\frac{1}{2}$ at $x = \frac{1}{n}$; $\frac{1}{n}$ tending to 0 as

$n \rightarrow \infty$. Let us take an interval $[a, b]$ containing 0.

Thus

$$\begin{aligned}
M_n &= \sup_{x \in [a, b]} |f_n(x) - f(x)| \\
&= \sup_{x \in [a, b]} \left| \frac{nx}{1 + n^2 x^2} \right| \\
&= \frac{1}{2}, \text{ which does not tend to zero as } n \rightarrow \infty.
\end{aligned}$$

Hence the sequence $\{f_n\}$ is not uniformly convergent in any interval containing the origin.

Theorem 4. (Weierstrass's M-test). A series of functions $\sum f_n$ will converge uniformly (and absolutely) on $[a, b]$ if there exists a convergent series $\sum M_n$ of positive numbers such that for all $x \in [a, b]$

$$|f_n(x)| \leq M_n, \text{ for all } n$$

Let $\varepsilon > 0$ be a positive number.

Since $\sum M_n$ is convergent, therefore there exists a positive integer N such that

$$|M_{n+1} + M_{n+2} + \dots + M_{n+p}| < \varepsilon \quad \forall n \geq N, p \geq 1 \quad \dots(1)$$

Hence for all $x \in [a, b]$ and for all $n \geq N, p \geq 1$, we have

$$|f_{n+1}(x) + f_{n+2}(x) + \dots + f_{n+p}(x)| \leq |f_{n+1}(x)| + |f_{n+2}(x)| + \dots + |f_{n+p}(x)| \quad \dots(2)$$

$$\leq M_{n+1} + M_{n+2} + \dots + M_{n+p}$$

$$< \varepsilon \quad \dots(3)$$

(2) and (3) imply that $\sum f_n$ is uniformly and absolutely convergent on $[a, b]$.

Example. Test for uniform convergence the series.

$$(i) \quad \sum \frac{x}{(n + x^2)^2}, \quad (ii) \quad \sum \frac{x}{n(1 + nx^2)}$$

Solution. (i) Here $u_n(x) = \frac{x}{(n + x^2)^2}$.

Now $u_n(x)$ is maximum or minimum when $\frac{du_n(x)}{dx} = 0$

$$\text{or} \quad (n+x^2)^2 - 4x^2(n+x^2) = 0$$

$$3x^4 + 2nx^2 - n^2 = 0$$

$$\text{or} \quad x^2 = \frac{n}{3} \text{ i.e. } x = \sqrt{\frac{n}{3}}.$$

If will be seen that $\frac{d^2u_n(x)}{dx^2}$ is -ve when $x = \sqrt{\frac{n}{3}}$.

$$\text{Hence Max } u_n(x) = \frac{\sqrt{\frac{n}{3}}}{\left(n + \frac{n}{3}\right)^2} = \frac{3\sqrt{3}}{16n^{3/2}} = M_n.$$

Therefore $|u_n(x)| \leq M_n$.

But $\sum M_n$ is convergent.

Hence the given series is uniformly convergent for all values of x by Weierstrass's M test.

(ii) Here $u_n(x)$ is Maximum or minimum when

$$n(1+nx^2) - 2n^2x^2 = 0 \text{ or } x = \pm 1/\sqrt{(n)}.$$

It can be easily shown that $x = \frac{1}{\sqrt{n}}$ makes $u_n(x)$ a maximum.

$$\text{Hence Max } u_n(x) = \frac{1/\sqrt{n}}{n(1+1)} = \frac{1}{2.n^{3/2}} = M_n. \text{ But } \sum M_n \text{ is convergent.}$$

Hence the given series is uniformly convergent for all values of x by Weierstrass's M-test.

Example :- Consider $\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$, $x \in \mathbf{R}$.

We assume that x is +ve, for if x is negative, we can change signs of all the terms.

We have

$$f_n(x) = \frac{x}{n(1+nx^2)}$$

and $f'_n(x) = 0$

implies $nx^2 = 1$. Thus maximum value of $f_n(x)$ is $\frac{1}{2n^{3/2}}$

Hence $f_n(x) \leq \frac{1}{2n^{3/2}}$

Since $\sum \frac{1}{n^{3/2}}$ is convergent, Weierstrass' M-Test implies that $\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$ is

uniformly convergent for all $x \in \mathbf{R}$.

Example :- Consider the series $\sum_{n=1}^{\infty} \frac{x}{(n+x^2)^2}$, $x \in \mathbf{R}$. We have

$$f'_n(x) = \frac{x}{(n+x^2)^2}$$

and so $f_n(x) = \frac{(n+x^2)^2 - 2x(n+x^2)2x}{(n+x^2)^4}$

Thus $f'_n(x) = 0$ gives

$$x^4 + x^2 + 2nx^2 - 4nx^2 - 4x^4 = 0$$

$$-3x^4 - 2nx^2 + n^2 = 0$$

or $3x^4 + 2nx^2 - n^2 = 0$

or $x^2 = \frac{n}{3}$ or $x = \sqrt{\frac{n}{3}}$

Also $f''_n(x)$ is -ve. Hence maximum value of $f_n(x)$ is $\frac{3\sqrt{3}}{16n^2}$. Since $\sum \frac{1}{n^2}$ is

convergent, it follows by Weierstrass's M-Test that the given series is uniformly convergent.

Example . The series $\sum \frac{x}{n^p + x^2 n^q}$ converges uniformly over any finite interval [a,

b], for (i) $p > 1, q \geq 0$ (ii) $0 < p \leq 1, p + q > 2$

(i) When $p > 1, q \geq 0$

$$|f_n(x)| = \left| \frac{x}{n^p + x^2 n^q} \right| \leq \frac{\alpha}{n^p}$$

where $\alpha \geq \max \{|a|, |b|\}$.

The series $\sum(\alpha / n^p)$ converges for $p > 1$.

Hence by M-test, the given series converges uniformly over the interval [a, b].

(ii) When $0 < p \leq 1, p + q > 2$.

$|f_n(x)|$ attains the maximum value $\frac{1}{2n^{\frac{1}{2}(p+q)}}$ at the point, where $x^2 n^q = n^p$

$$\therefore |f_n(x)| \leq \frac{1}{2n^{\frac{1}{2}(p+q)}}$$

The series $\sum \frac{1}{2n^{\frac{1}{2}(p+q)}}$ converges for $p + q > 2$. Hence by M-test, the given series

converges uniformly over any finite interval [a, b].

Example . Test for uniform convergence, the series

$$\frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + \dots, -\frac{1}{2} \leq x \leq \frac{1}{2}$$

Solution. The nth term $f_n(x) = \frac{2^n x^{2^n-1}}{1+x^{2^n}}$

$$|f_n(x)| \leq 2^n (\alpha)^{2^n-1}$$

where $|x| \leq \alpha \leq \frac{1}{2}$.

The series $\sum 2^n (\alpha)^{2^n-1}$ converges, and hence by M-test the given series converges uniformly on $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

