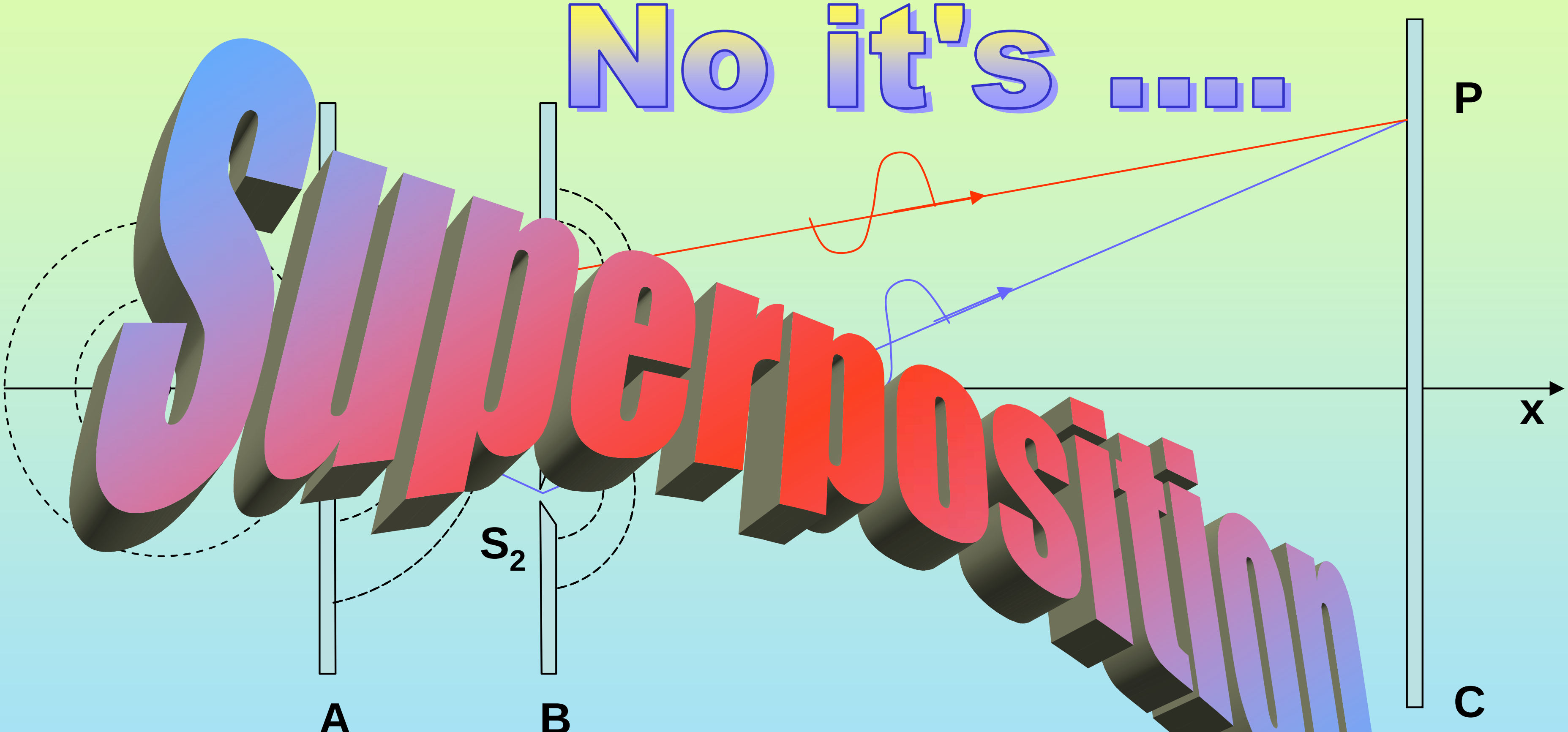
It's a bird! It's a plane!

No it's ....

Superposition



another Physics presentation

Superposition of Waves:

As we have seen previously, the defining property of a wave is that it can be described by a wave function of the form -

$y = F(x - vt)$	for any wave
$y = a \sin k(x - vt)$	for an harmonic wave

where, y is a wave displacement.

For string waves, y is the mechanical displacement of a string particle from its centre position; for sound waves in a gas, y could be either the mechanical displacement of a molecule from its centre position, or an associated gas pressure difference from ambient pressure; for radio waves, the displacement would be an electric, or magnetic field vector.

When two different waves, of the same type, pass, simultaneously, through the same point in space, the individual wave displacements add -

$$y = y_1 + y_2 \quad \text{---------------(1)}$$

where y_1 = the wave displacement at the point due to the first wave only.

y_2 = the wave displacement at the point due to the second wave only

and y = the total resultant wave displacement at the point due to both waves

Equation (1) is called the **"principle of superposition".**

Linear Superposition:

$$y = y_1 + y_2 \text{ ---------------(1)}$$

Waves propagate through media, by deforming the medium. If the amplitude of a wave is small, this deformation will be within the elastic limits of the medium, and proportional to the wave forces causing it. When more than one such wave passes a point in such a medium, their individual amplitudes add linearly, as in equation (1). However, in extreme cases, where wave amplitudes are large, the elastic limit of the medium can be exceeded, and amplitudes will add non-linearly (equation (1) does not hold). We will consider only the **linear superposition** case (equation (1)).

We will assume the superposing waves to be harmonic (sinusoidal). These waves can differ in amplitude, frequency, wave velocity, and direction of travel.

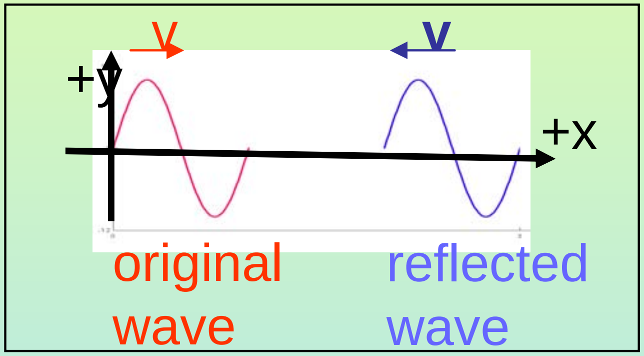
Whilst any two waves can superpose, the most important cases are where the two waves are identical, except in **one wave parameter**. We will deal with the following cases, which have the following single difference -

1. Standing Waves	waves travel in opposite directions
2 Beats	waves have slightly different frequencies
3. Interference	waves differ in phase

1.Standing Waves - Introduction:

This is our first case of superposition.

Standing waves occur, when two identical waves traverse a medium in opposite directions. This happens, when a wave is continuously reflected back along its path, so that the original and reflected waves pass through each other.



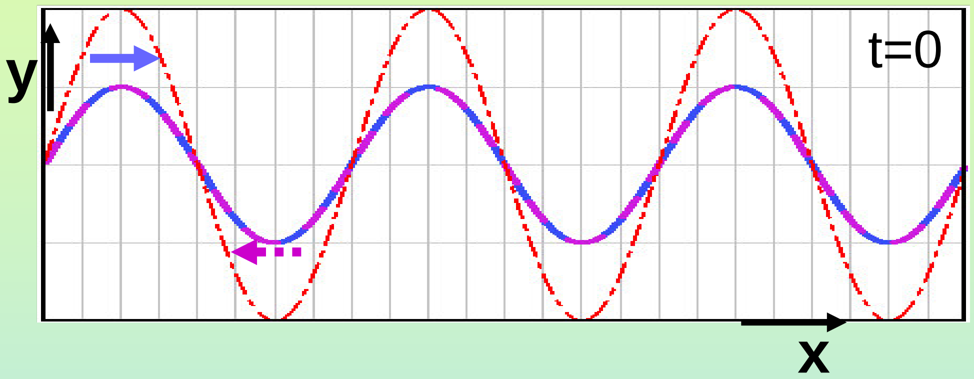
Examples:

In **stringed musical instruments**, such a piano, guitar, or violin, a string is plucked, or otherwise disturbed, so that a wave propagates along the string. On reaching the end of the string, this wave is reflected back, so that a standing wave is established on the string. The standing wave is the source of the musical note. Wind instruments are similar, except that the wave is a sound wave in air.

In **lasers** a light wave is sent through a cavity bounded by two mirrors. The light wave reflects back and forth between the two mirrors, so an optical standing wave is set up in the cavity. The properties of the laser light depend largely on this standing wave.

Standing Waves - Graphical :

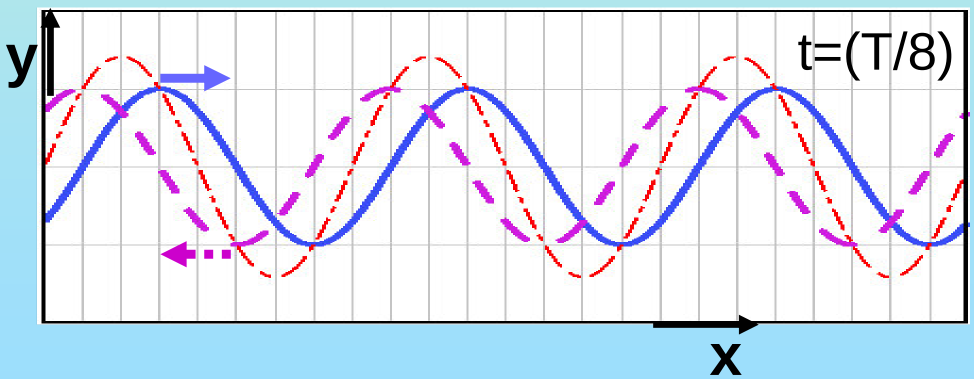
To understand standing waves, we will look at the details of their development, graphically. We start our clock (t = 0), when the two oppositely-traveling waves are in phase with each other. Their sum (the standing wave) is shown in red (– · – · ·).



+x direction
-x direction
sum wave

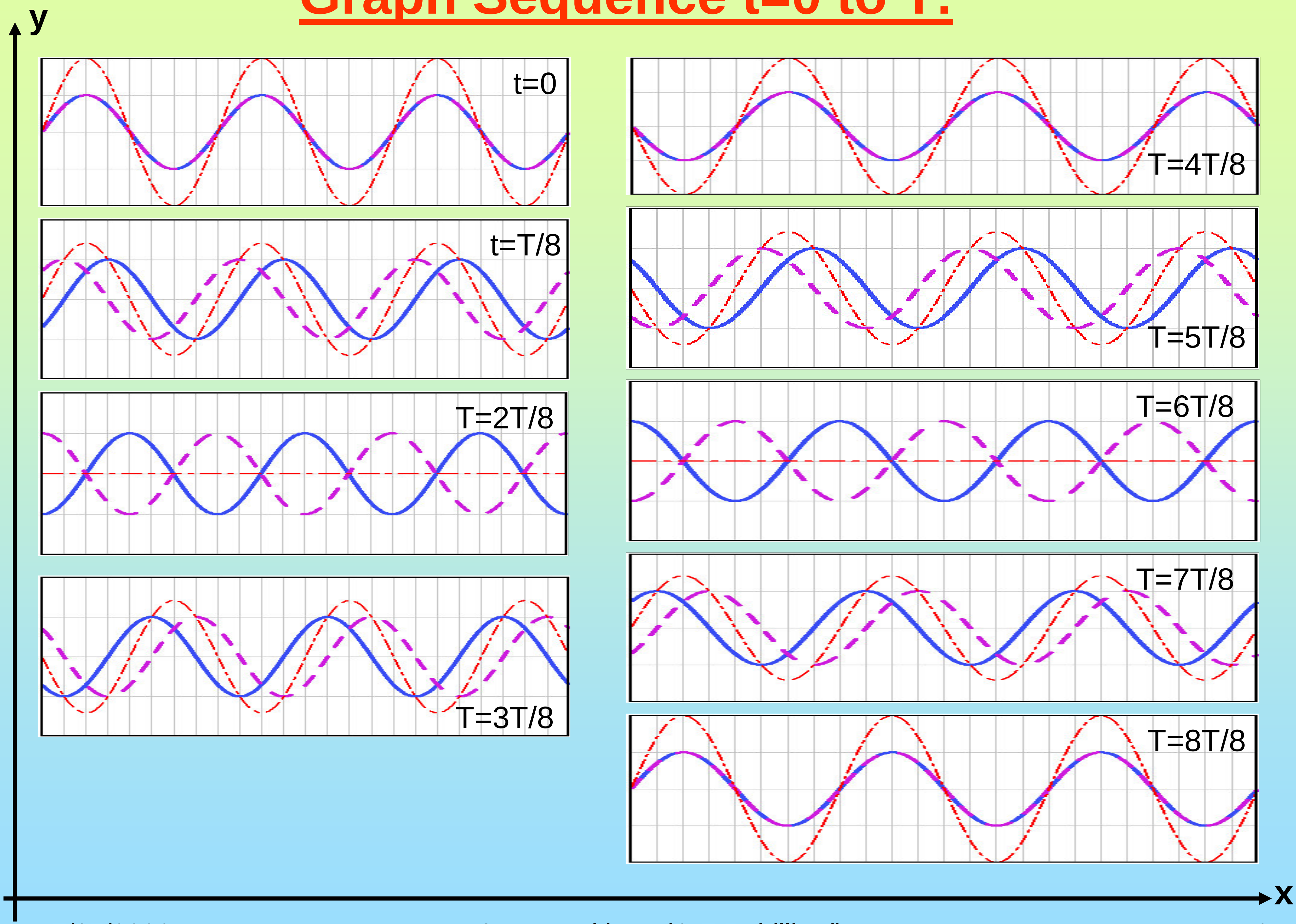
Let T = period, and λ = wavelength of each of the traveling waves.

Time (T/8) later, the blue (——) wave has moved distance (λ/8) in the (+x)-direction, and the purple (– – –) wave has traveled (λ/8) in the (–x)-direction. The red sum wave (– · – · ·) has not moved, but its magnitude has been reduced.



The full development of the standing wave, over a time T, is shown in the next slide.

Graph Sequence t=0 to T:

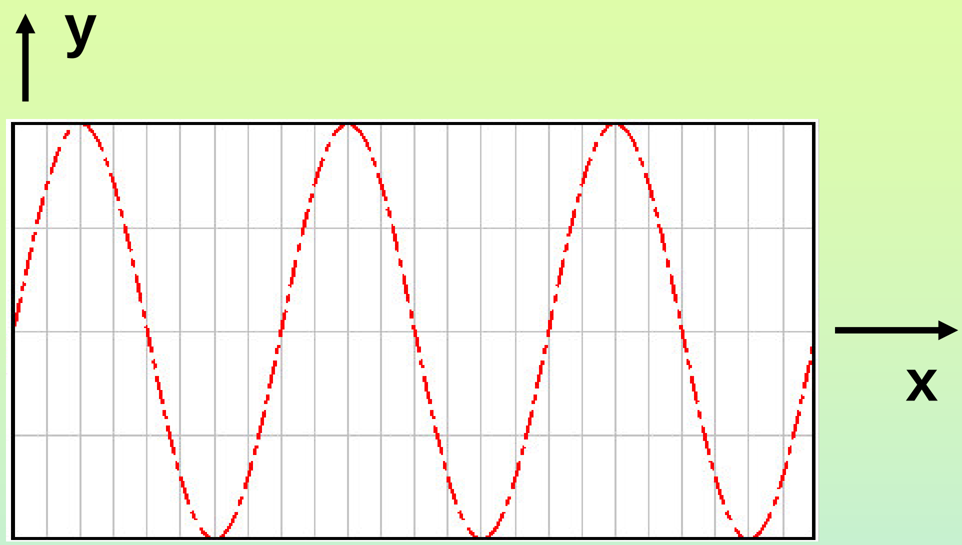


Standing Wave Only:

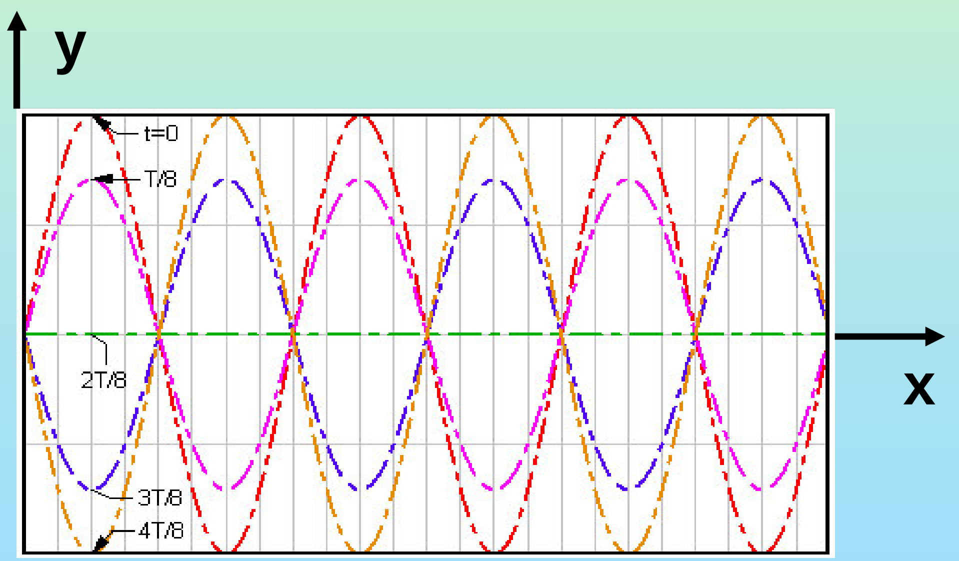
Let us extract the **standing wave** (the sum wave), from the graphs of the previous slides, over a time interval

t = 0 to T/2.

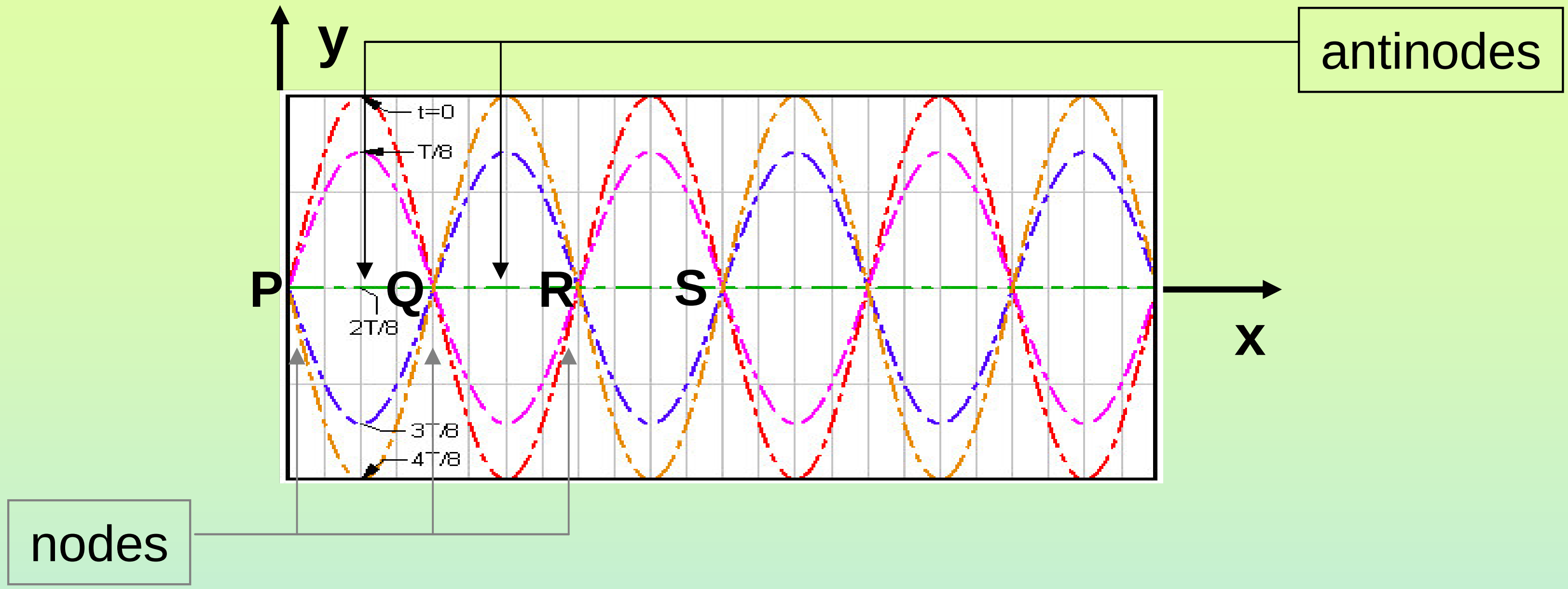
(**Note**: this animation will not render in pdf)



Drawing all these sum waves onto the same graph, using different colours:



Nodes & Antinodes:



Along the x-axis, there are points, such as P, Q, R, & S, where the amplitude of vibration is zero. These points are called **nodes**.

Points midway between nodes have a maximum amplitude. These points are called **antinodes**.

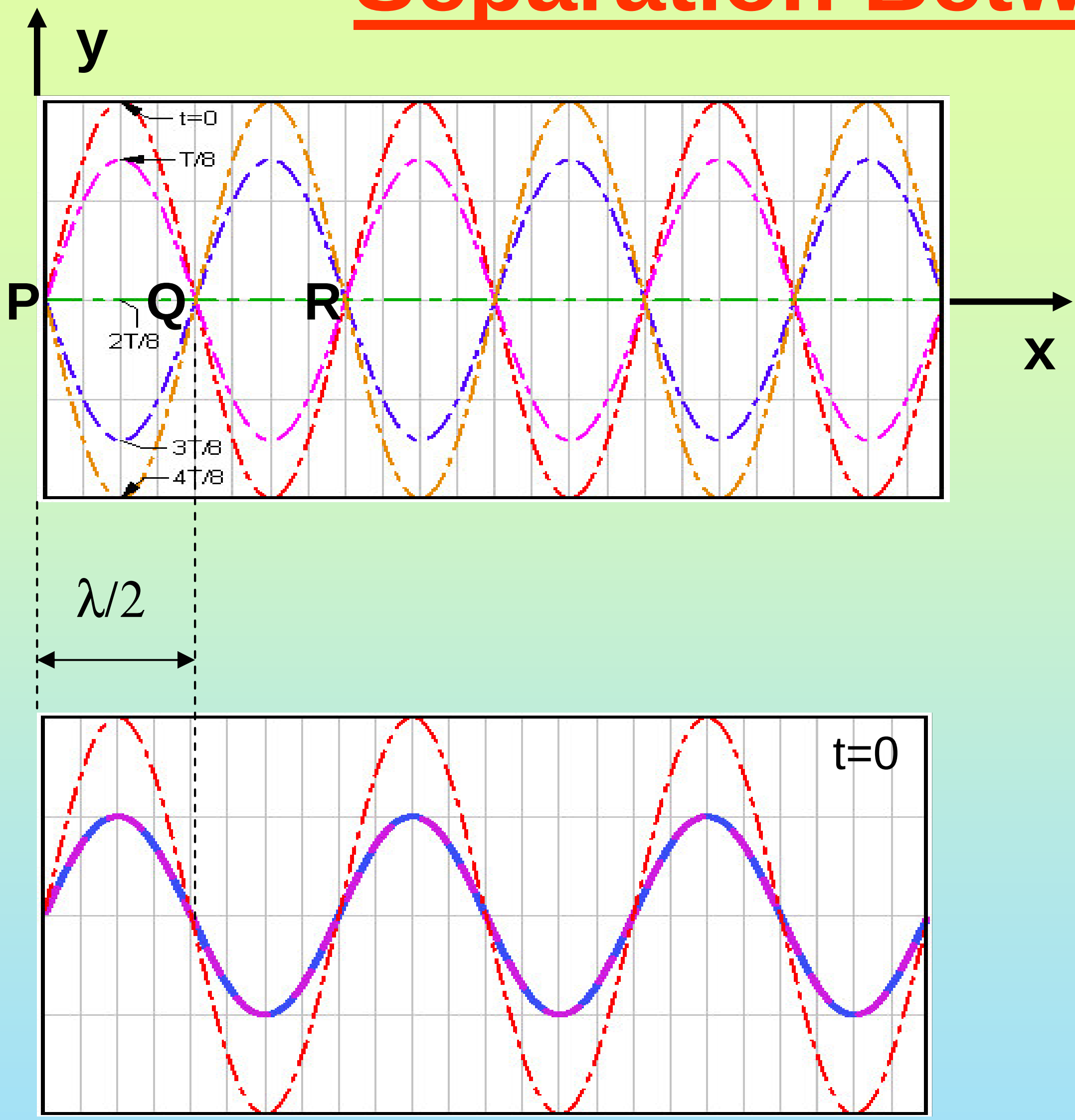
Between adjacent nodes (eg.between P & Q), all points move together – are in phase.

Points between P & Q, are 180 deg out of phase with points between Q & R.

Points between P & Q, are in phase with points between R & S.

These waves are called “standing”, or “stationary” because their node, and antinode positions remain fixed.

Separation Between Nodes :

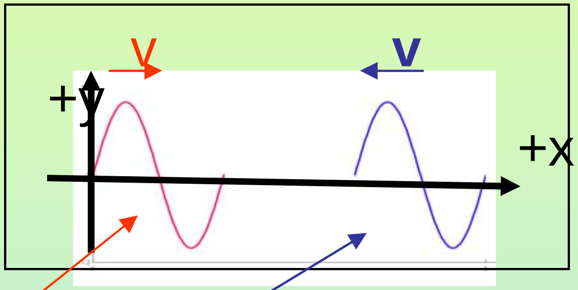


By comparing the standing wave, with the two component traveling waves, it is clear, that the distance between adjacent nodes is half a wavelength of the traveling waves = $\lambda/2$.

Standing Waves – Wave Function:

Next we develop the wave function for a standing wave.

Standing waves occur, when two identical waves traverse a medium in opposite directions. We will assume harmonic waves.



For both waves: a = amplitude
k = wave number
ω = angular frequency

We firstly write the wave function for each traveling wave

If the initial wave is: $y_1 = a \sin(kx - \omega t)$ ------(1)

the reflected wave will be: $y_2 = a \sin(kx + \omega t)$ ------(2)

where y_1 and y_2 are the individual displacements for each wave.

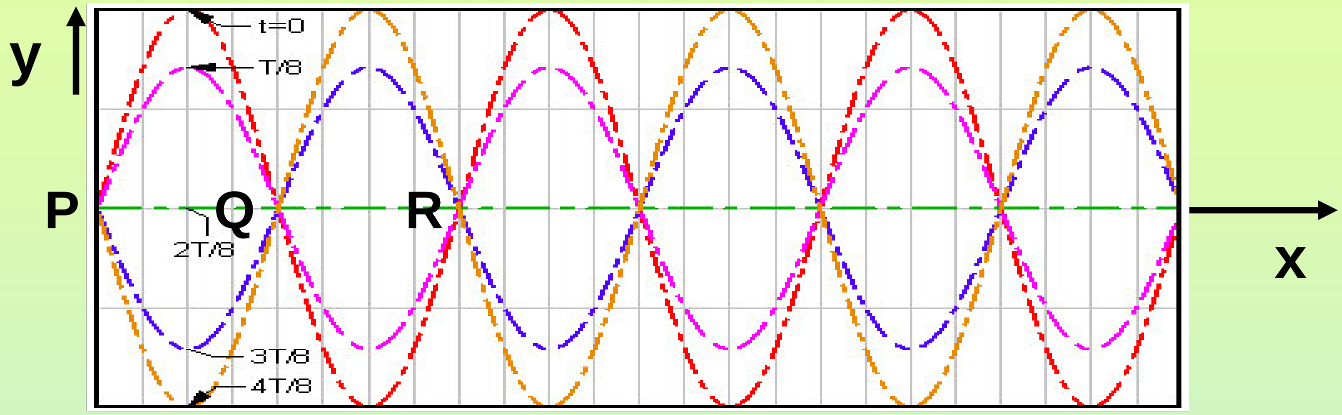
At any point, x, in the medium, at any time, t, the resultant displacement, y, will be:

$$y = y_1 + y_2$$
$$= a \sin(kx - \omega t) + a \sin(kx + \omega t)$$
$$= 2a \sin(kx) \, . \cos(\omega t)$$

since: $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

Standing-Wave Wave Function: $y = [2a \sin(kx)] . \cos(\omega t)$ ---(3)

Interpretation of Wave Function:



Standing-Wave Wave Function: $y = [2a \sin(kx)].\cos(\omega t)$ ---(3)

If we fix t, (3) becomes $y = [2a \cos(\omega t)]\ \sin(kx) = A \sin(kx)$
where $A = [2a \cos(\omega t)]$ acts as the amplitude of the standing wave pattern, at particular moments, as represented graphically, in the individual sine curves plotted above.

If we fix x, (3) becomes $y = [2a \sin(kx)]\ \cos(\omega t)] = A' \cos(\omega t)$
where $A' = [2a \sin(kx)]$ acts as an amplitude term, giving the amplitude of vibration of the particular particle located at the given x value.

Node Positions from Wave Function:

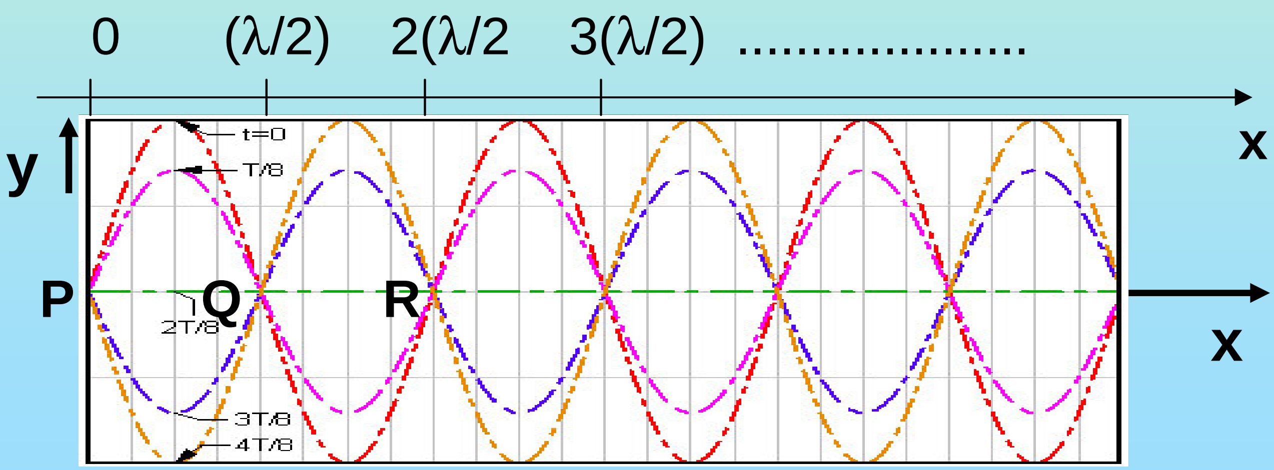
We can find the positions of nodes, from the wave function.

Standing-Wave Wave Function: $y = [2a \sin(kx)].\cos(\omega t) = A \cos(\omega t)$ -----(3)

At nodes, the amplitude of vibration of particles is zero, at all times, t:

$$A = [2a \sin(kx)] = 0$$
$$\therefore \ (kx) = 0, \pi, 2\pi, 3\pi, ...$$
$$\therefore \quad x = 0, \left(\frac{\lambda}{2}\right), 2\left(\frac{\lambda}{2}\right), 3\left(\frac{\lambda}{2}\right), ... \text{ since } k \equiv \frac{2\pi}{\lambda}$$

Nodes are equally spaced, and separated by a distance of (λ/2), where λ is the wavelength of either of the component traveling waves.



Which corresponds to our previous graphical result.

Antinode Positions from Wave Function:

We can also find the positions of antinodes, from the wave function.

Standing-Wave Wave Function: $y = [2a \sin(kx)].\cos(\omega t) = A \cos(\omega t)$ -----(3)

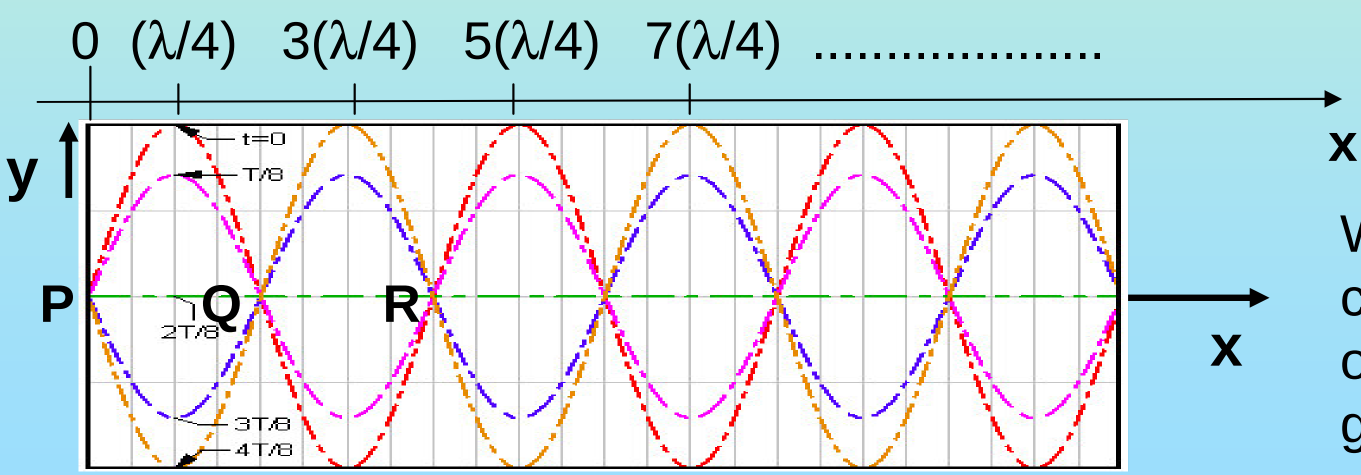
At antinodes, the amplitude of vibration of particles is max., at all times, t:

$$A = [2a \sin(kx)] = 2a \rightarrow \sin(kx) = 1$$

$$\therefore (kx) = \left(\frac{\pi}{2}\right), 3\left(\frac{\pi}{2}\right), 5\left(\frac{\pi}{2}\right), 7\left(\frac{\pi}{2}\right), ...$$

$$\therefore \quad x = \left(\frac{\lambda}{4}\right), 3\left(\frac{\lambda}{4}\right), 5\left(\frac{\lambda}{4}\right), 7\left(\frac{\lambda}{4}\right), ... \text{ since } k \equiv \frac{2\pi}{\lambda}$$

Antinodes are equally spaced, separated by a distance of (λ/2), and located midway between the nodes.



Which also corresponds to our previous graphical result.

Example:

A standing wave is represented by the following wave function:

$$y = 0.004 \sin(100x).\cos(300t) \qquad \text{(SI units)}$$

Find the amplitude, frequency, wavelength, and velocity of the component traveling waves. Find also, the separation of nodes.

Compare this wave-function with the general case.

General case: $y = \boxed{2a} \sin \boxed{(kx)} \cos \boxed{(\omega t)}$ ----------------------(3)

This wave: $y = \boxed{0.0040} \sin \boxed{(100x)}. \cos \boxed{(600t)}$ --------------------(4)

Amplitude:
2a = 0.0040
a = 0.0020 m
=2.0 mm

Wavelength:
$k = 100$
$(2\pi/\lambda) = 100$
$\lambda = (2\pi)/100$
$= 0.063$ m
$= 6.3$ cm

Frequency:
$\omega = 600$
$2\pi f = 600$
$f = 600/(2\pi)$
$= 96$ Hz

Velocity of traveling waves = $f\lambda$ = (96) (0.063) = 6.05 m/s

Separation of nodes = $\lambda/2$ = (6.3)/2 = 3.1 cm

Modes of Vibration:

Here, we will consider finite media. These are media, that have boundaries. Examples are the strings of musical instruments, which are bounded by their ends, the bounded air columns within wind instruments, and the membrane of a drum, which is bounded by its rim.

When a bounded medium is vibrated by some external force, waves are set up in the medium. These waves are reflected back from the boundaries. The interaction of the waves and their reflections produce stationary waves in the medium.

However, conditions at the boundary will determine the vibration at the boundary. For example, the boundary may be rigid, so that the vibration there must have zero amplitude, and must consequently be a displacement **node**.

Such boundary conditions limit the ways in which standing waves can be set up in the medium, and therefore limit the ways in which the medium can vibrate.

The discrete ways in which a bounded medium can vibrate are called its **modes of vibration**.

Each of these modes will be associated with certain standing wave frequencies.

Thus, there will be only certain, characteristic, discrete frequencies of vibration possible – other frequencies will be suppressed.

Stretched Strings:

This is our first example of a bounded vibrating medium.

Assume that the string is stretched between two rigid support points. An oscillating force acts on the string so as to cause standing waves to be set up on the string.

Because the string is **fixed at both ends, both ends must be nodes**. Only standing waves that comply with this requirement can be established on the string. Each of these standing waves corresponds to a vibratory mode of the string.

Let the various modes be characterised by a mode number, i, where i = 1, 2, 3, 4, 5, ...

Let L = length of the string
v = speed of the traveling waves along the string
λ_i = the wavelength of the i th mode of vibration of the string
= the wavelength of the component traveling waves for the i th mode
f_i = the frequency of the i th mode of vibration
= the frequency of the component traveling waves for the i th mode

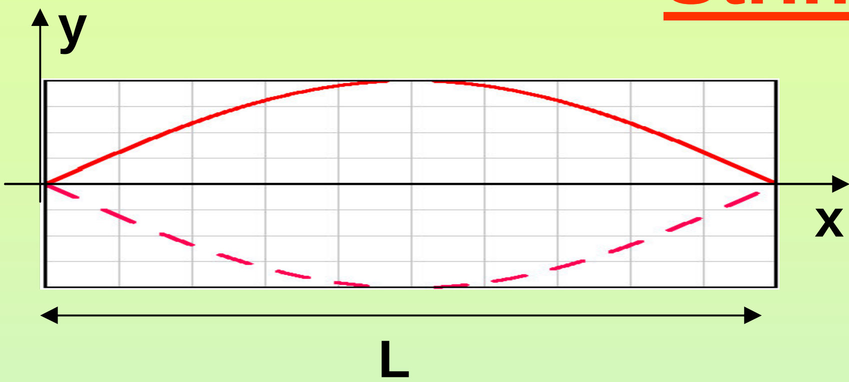
Remember: for traveling waves $v = f\,\lambda$

thus: for the traveling waves forming the i th mode standing wave

$v = f_i\,\lambda_i$

therefore: $f_i = f/\lambda$

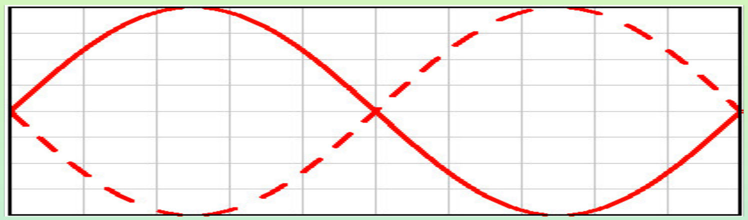
String Modes:



Fundamental (1st) mode

$$L = \left(\frac{\lambda_1}{2}\right) \rightarrow \lambda_1 = 2L$$

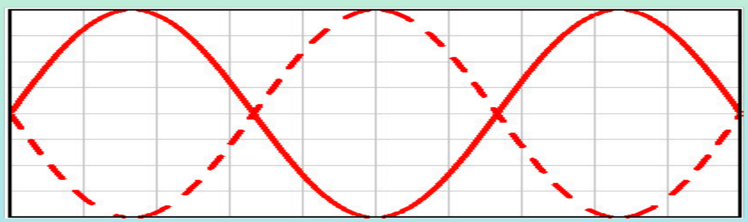
$$f_1 = \frac{v}{\lambda_1} = \left(\frac{v}{2L}\right) \;.......(1)$$



2nd vibratory mode

$$L = 2\left(\frac{\lambda_2}{2}\right) \rightarrow \lambda_2 = L$$

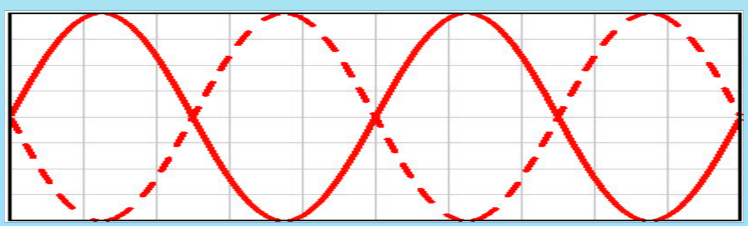
$$f_2 = \frac{v}{\lambda_2} = \left(\frac{v}{L}\right) = 2\,f_1 \;.......(2)$$



3rd vibratory mode

$$L = 3\left(\frac{\lambda_3}{2}\right) \rightarrow \lambda_3 = \frac{2}{3}L$$

$$f_1 = \frac{v}{\lambda_3} = 3\left(\frac{v}{2L}\right) = 3\,f_1 \;.......(3)$$



4th vibratory mode

$$L = 4\left(\frac{\lambda_4}{2}\right) \rightarrow \lambda_4 = \frac{L}{2}$$

$$f_4 = \frac{v}{\lambda_4} = 4\left(\frac{v}{2L}\right) = 4\,f_1 \;.......(4)$$

Modal frequencies:

The string can vibrate in many modes, each one of which has a characteristic frequency. The i th mode has frequency f_i

The simplest mode (fewest nodes) has the lowest frequency, f_1, which is called the **fundamental frequency** of the string. Higher modes have frequencies that are multiples of this fundamental.

The fundamental frequency, f_1, of a stretched string is given by:

$$f_1 = \left(\frac{v}{2L}\right) \qquad \text{from } (1)$$

$$= \frac{1}{2L}\sqrt{\frac{T}{\mu}} \qquad \text{since } v = \sqrt{\frac{T}{\mu}}$$

T = tension in string

m = string's mass per unit length

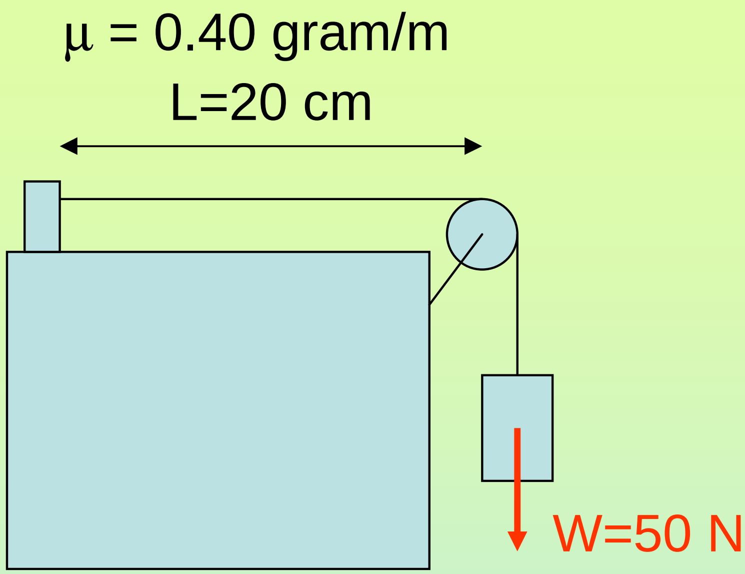
For a given string, under fixed tension, the frequency of the fundamental is inversely proportional to the length of the string.

From equations (1), (2), (3), (4), the modal frequencies for the string are in the ratios:

$$f_1 : f_2 : f_3 : f_4 \ \ldots\ldots\ldots = 1:2:3:4 \ldots\ldots\ldots$$

$$f_n = n\frac{v}{\lambda_1} = n\left(\frac{v}{2L}\right) = n\, f_1 \quad (n = 1,2,3\ldots\ldots)$$

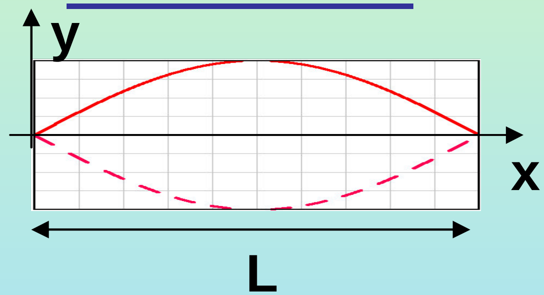
When excited, the string will **vibrate simultaneously in all these possible modes**. However, amplitude of the mode decreases, as their frequency increases.



Example:

A mass of weight = W = 50 N is hung from the end of a string of mass per unit length = m = 0.40 gram/metre, as shown. Find the fundamental frequency with which the string will vibrate, when plucked. Find also, the frequencies of the second, and third vibrational modes.

Fundamental:



$$L = \left(\frac{\lambda_1}{2}\right) \rightarrow \lambda_1 = 2L$$

$$\therefore f_1 = \frac{v}{\lambda_1} = \left(\frac{v}{2L}\right) = \frac{1}{2L}\sqrt{\frac{T}{\mu}} = \frac{1}{2\times 0.20}\sqrt{\frac{50}{0.40\times 10^{-3}}} = 884\ \mathrm{Hz}$$

Second & third modes:

$$f_1 = 884\ \mathrm{Hz} \quad \therefore \quad f_2 = 2x884 = 1.77\ \mathrm{kHz}$$

$$\& \quad f_3 = 3x884 = 2.65\ \mathrm{kHz}$$

Fundamental, second and third modes have frequencies 0.884, 1.77, & 2.65 kHz

Acoustic cavities:

This is our second example a bounded medium.
The medium is a gas which is bounded by the ends of the cavity which contains it.

When the cavity is excited by some vibrating external force, the resultant traveling sound waves are reflected from the ends of the cavity, producing standing waves.

The end of an acoustic cavity can be **closed** (a solid boundary), or **open** (no boundary)

If the end is **closed**, the displacement of air particles, there, must be zero, constituting a displacement **node**.

If the end is **open**, the vibrating air particles there, have maximum freedom to vibrate, and the end is a displacement **antinode**.

The particular boundary conditions will determine the standing wave patterns, that can be established in the cavity. Each of these patterns is a vibrational mode of the cavity, and each will have a characteristic frequency.

There are three cases, depending on whether the two ends are open or closed:.

Closed-Open **Open-Open** **Closed-Closed**

Let: L = the cavity length. Other terms are as for stretched strings.

Closed-Open:

Fundamental mode

$$\lambda_1 = 4L$$

$$f_1 = \frac{v}{\lambda_1} = \left(\frac{v}{4L}\right) \quad \ldots\ldots\ (4)$$

2nd vibratory mode

$$\lambda_2 = \frac{4}{3}L$$

$$f_2 = \frac{v}{\lambda_2} = 3\left(\frac{v}{4L}\right) = 3\,f_1 \quad \ldots\ldots\ (5)$$

3rd vibratory mode

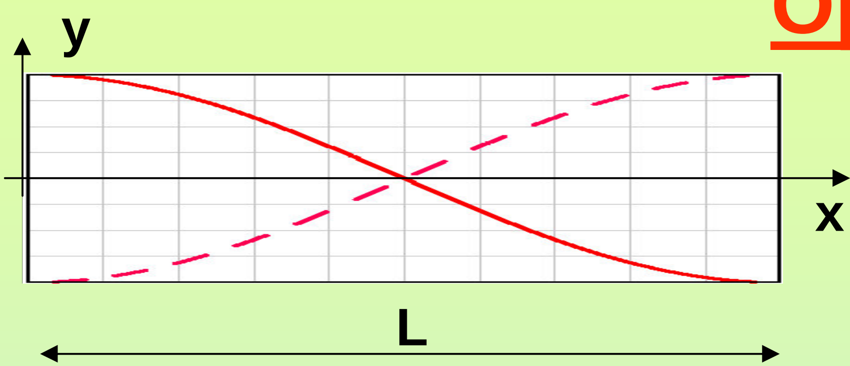
$$\lambda_3 = \frac{4}{5}L$$

$$f_3 = \frac{v}{\lambda_3} = 5\left(\frac{v}{4L}\right) = 5\,f_1 \quad \ldots\ldots\ (6)$$

$$f_n = n\frac{v}{\lambda_1} = n\left(\frac{v}{4L}\right) = n\,f_1 \quad (n = 1,3,5\ldots\ldots)$$

Mode frequencies are in ratio $f_1:f_2:f_3 \ldots\ldots = 1:3:5\ldots\ldots$

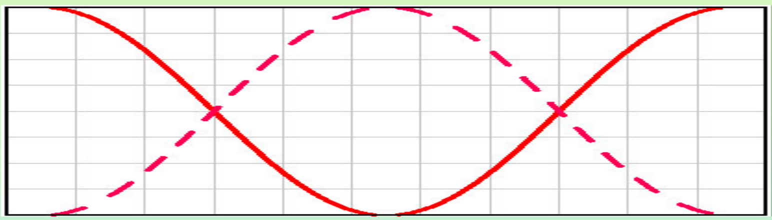
Open-Open:



Fundamental (1st) mode

$$\lambda_1 = 2L$$

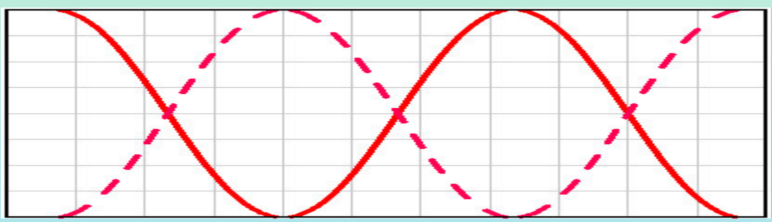
$$f_1 = \frac{v}{\lambda_1} = \left(\frac{v}{2L}\right) \quad ....(7)$$



2nd vibratory mode

$$\lambda_2 = L$$

$$f_2 = \frac{v}{\lambda_2} = 2\left(\frac{v}{2L}\right) = 2\,f_1 \quad ....(8)$$



3rd vibratory mode

$$\lambda_3 = \frac{2}{3}L$$

$$f_3 = \frac{v}{\lambda_3} = 3\left(\frac{v}{2L}\right) = 3\,f_1 \quad ....(9)$$

$$f_n = n\frac{v}{\lambda_1} = n\left(\frac{v}{2L}\right) = n\,f_1 \quad (n = 1,2,3.....)$$

Mode frequencies are in ratio $f_1{:}f_2{:}f_3$ ..... = 1:2:3......

Closed-Closed:

When both ends of the acoustic cavity are closed, the modal patterns set up correspond to those of the stretched string. The modal frequencies are in the ratio 1:2:3.........

Nomenclature:

Fundamental:

The fundamental mode is the one with the smallest frequency.

Harmonic:

A harmonic is a frequency that is an integer multiple of the fundamental frequency. The i-th harmonic has a frequency i times the fundamental.

Overtone:

An overtone is any standing-wave frequency above the harmonic. The j-th overtone is the j-th mode above the fundamental.

Nomenclature:

To clarify notation, the numbers of the mode, harmonic, and overtone are tabled below, firstly for a 1:2:3 system, and secondly for a 1:3:5 system.
f_1 = fundamental frequency

1:2:3 system:

frequency	f_1	$2f_1$	$3f_1$	$4f_1$
mode	1	2	3	4
harmonic	1	2	3	4
overtone		1	2	3

Example:
3rd mode = 3rd harmonic = 2nd overtone

All harmonics are present

1:3:5 system:

frequency	f_1	$3f_1$	$5f_1$	$7f_1$
mode	1	2	3	4
harmonic	1	3	5	7
overtone		1	2	3

Example:
3rd mode = 5th harmonic = 2nd overtone

Only the odd harmonics are present – even harmonics are missing.

Physics of Music:

All acoustic musical instruments consist of a bounded medium, in which standing waves are set up. For example, in a violin, guitar, and piano, standing waves are set up on a string. In a flute, clarinet, and organ, standing waves are set up in an air column. In a drum, a 2-dimensional standing wave is set up on a membrane.

As we have seen, many standing wave modes, with increasing frequencies, occur simultaneously, but with decreasing amplitude, as the frequency of the mode increases. The total wave, in the medium, is the instantaneous sum of the many modes present.

A discrete sound played on a musical instrument is called a **note**.

The sound of a musical note, played on an instrument, is determined by its particular pattern of standing waves – the standing waves generate the sound.

There are three important properties of a musical note: **loudness**, **pitch**, and **tone colour**

Loudness:

Loudness (or volume) of a note is determined by the **amplitude** of the standing waves (in particular, the fundamental) producing it. This can be controlled by the vigor with which the player of the instrument generates the standing waves.

Pitch and Musical Scales:

The pitch of a musical note is determined by the **frequency of the fundamental.** Other harmonics are not recognized in the ear's determination of pitch. **For example:** "middle C" on the musical scale has a fundamental frequency of 261.63 Hz.

In music, notes are generally organized into pleasing sequences according to pitch. These sequences are called musical scales.

The fundamental frequencies of the notes in a musical scale have particular ratios to the fundamental frequency of the key note of that scale. There are several types of scales, depending on the particular set of ratios.

For example, the 8 notes, in the major key, of the Just (or Helmholtz) scale are in the following frequency ratios:

note nr. (n)	1	2	3	4	5	6	7	8
f_n / f_1	1.00	9/8 =1.12	5/4 =1.20	4/3 =1.33	3/2 =1.50	5/3 =1.60	15/8 =1.80	2.00

The n = 1 note is called the **key note** of the scale.

The C Major Scale :

note nr. (n)	1	2	3	4	5	6	7	8
f_n / f_1	1.00	9/8 =1.12	5/4 =1.20	4/3 =1.33	3/2 =1.50	5/3 =1.60	15/8 =1.80	2.00

If the key note is middle C (261.63 Hz = 262 Hz), the scale is C major, and the names given to the notes, and their frequencies are:

note name	C	D	E	F	G	A	B	C
f_n (Hz)	262	295	327	349	393	437	491	524

If the frequency of the note is a simple ratio to the frequency of the key note (n =1), then that note tends to be "consonant" (or pleasant when sounded) with the key note. n = 8 is the most consonant note, called the "octave", and is twice the frequency of the key note. The second most consonant note is n = 5, called the "dominant".

Tone Colour:

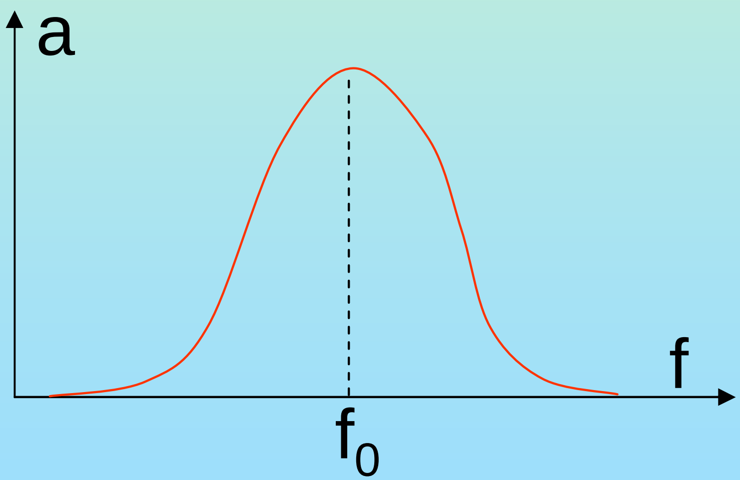
The characteristic sound ("tone colour") of a particular instrument is determined by the characteristic mixture of harmonics it produces, and their relative amplitudes. **For example:** middle C played on a violin sounds different to middle C played on a flute.

Resonance:

If a system, in which standing waves can be set up, is disturbed, and then left to oscillate by itself, it will oscillate only at the frequencies of its natural standing wave modes. These are the natural frequencies of the system.

However, if such a system is driven continuously, by an external periodic force, it can be forced to oscillate at other frequencies.

Let's say, that the frequency of the driving force (f) is varied, and the resulting amplitude of vibration of the system is measured. The amplitude (a) with which the system vibrates, will be greatest when the driving frequency equals, or is near, a natural frequency (f_0) of the system

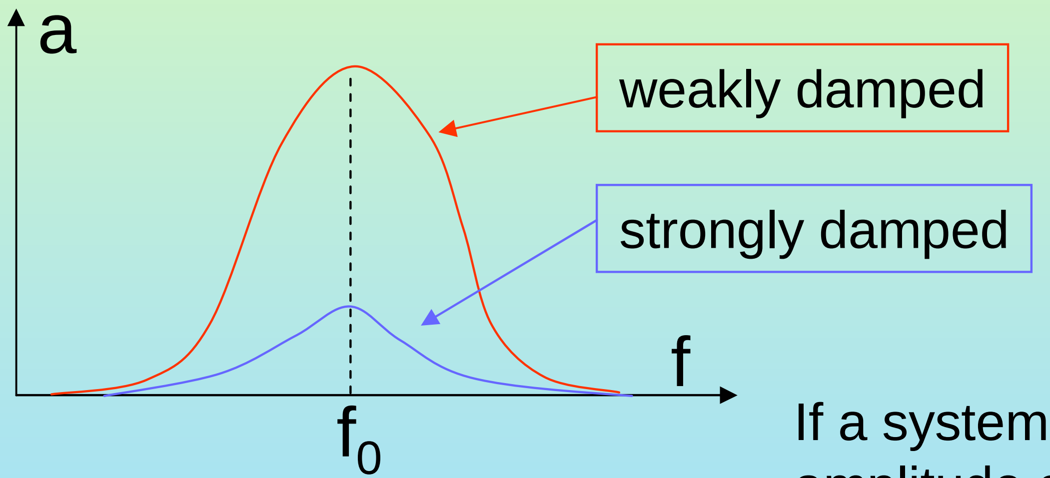


This phenomenon is called **resonance**, and the natural frequencies of vibration are the **resonant frequencies** of the system.

Damping:

When first driven at the resonant frequency, the system will absorb energy and, as a consequence, the amplitude of the system will increase. As the amplitude increases, losses due to friction increase, until all the input energy is dissipated in friction. At this point, the amplitude has reached its maximum.

We say that the system is **damped** by its internal friction. The greater the damping, the smaller will be the final resonant amplitude of the system.



Damping is often measured in terms of a quantity called the Q-factor of the system. Weakly damped systems have a high Q; strongly damped, a low Q.

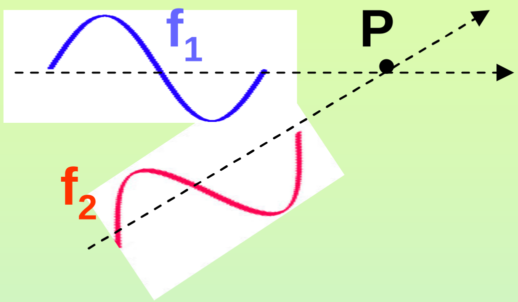
If a system is weakly damped, the large amplitude at resonance can lead to damage to the system – for example: shattering a champagne glass with a sustained note.

Examples of Resonance:

a child's swing, champagne glass, the tuned circuit of a radio receiver.

2.Beats:

This is our second case of superposition.



Beats occur when two waves, which are identical, except for a (slight) difference in frequency, pass through the same point.

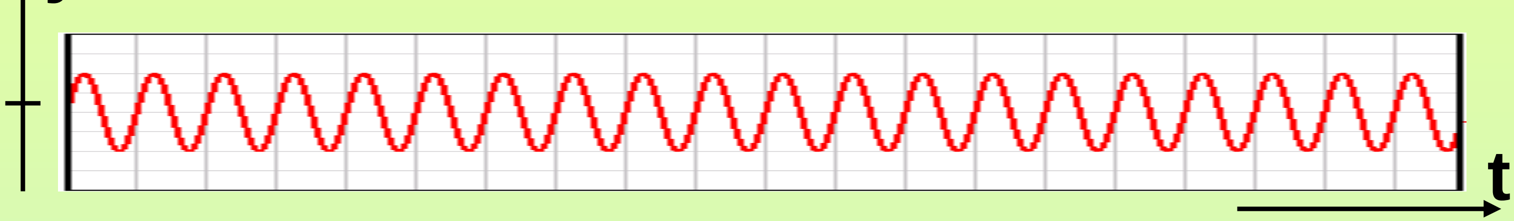
Say the two waves start in step. Because their frequencies are slightly different, the waves will progressively get further out of step, until they have a phase difference of 180 deg, and consequently annul each other, at the point. They will then progressively get back into step with each other, until they are back in phase, and consequently reinforce each other.

When the waves are in step, their amplitudes add, and a maximum occurs; when out of step, their amplitudes cancel, and a minimum results. If the waves are acoustic, this produces a waxing and waning in the volume of the resultant sound, which is called "beats".

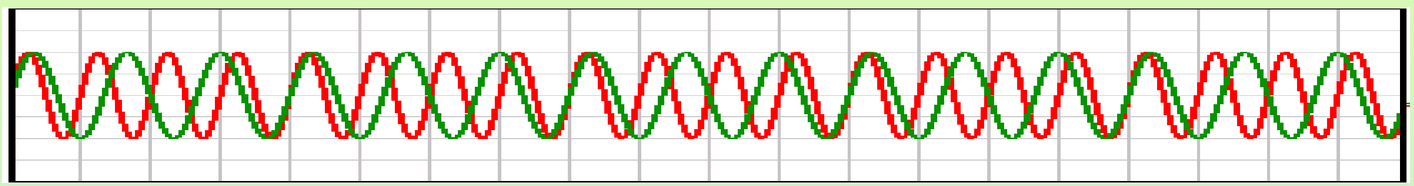
This can be seen graphically as follows. Note, that we are representing the displacements of the waves over time, at a fixed point in space.

Beats Graphically:

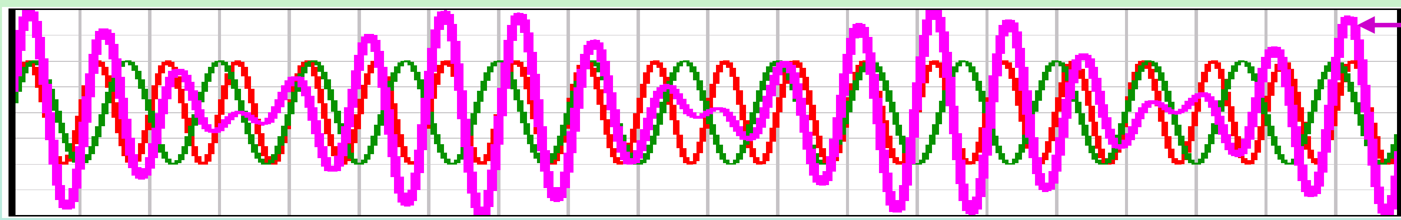
y



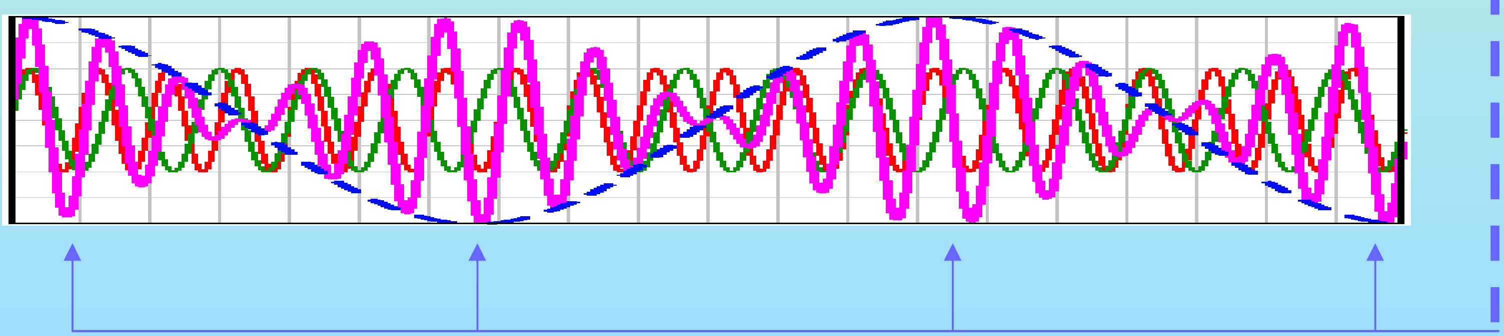
Displacement, y, at a point, P, due to single wave.



Individual displacements due to two waves of different frequency



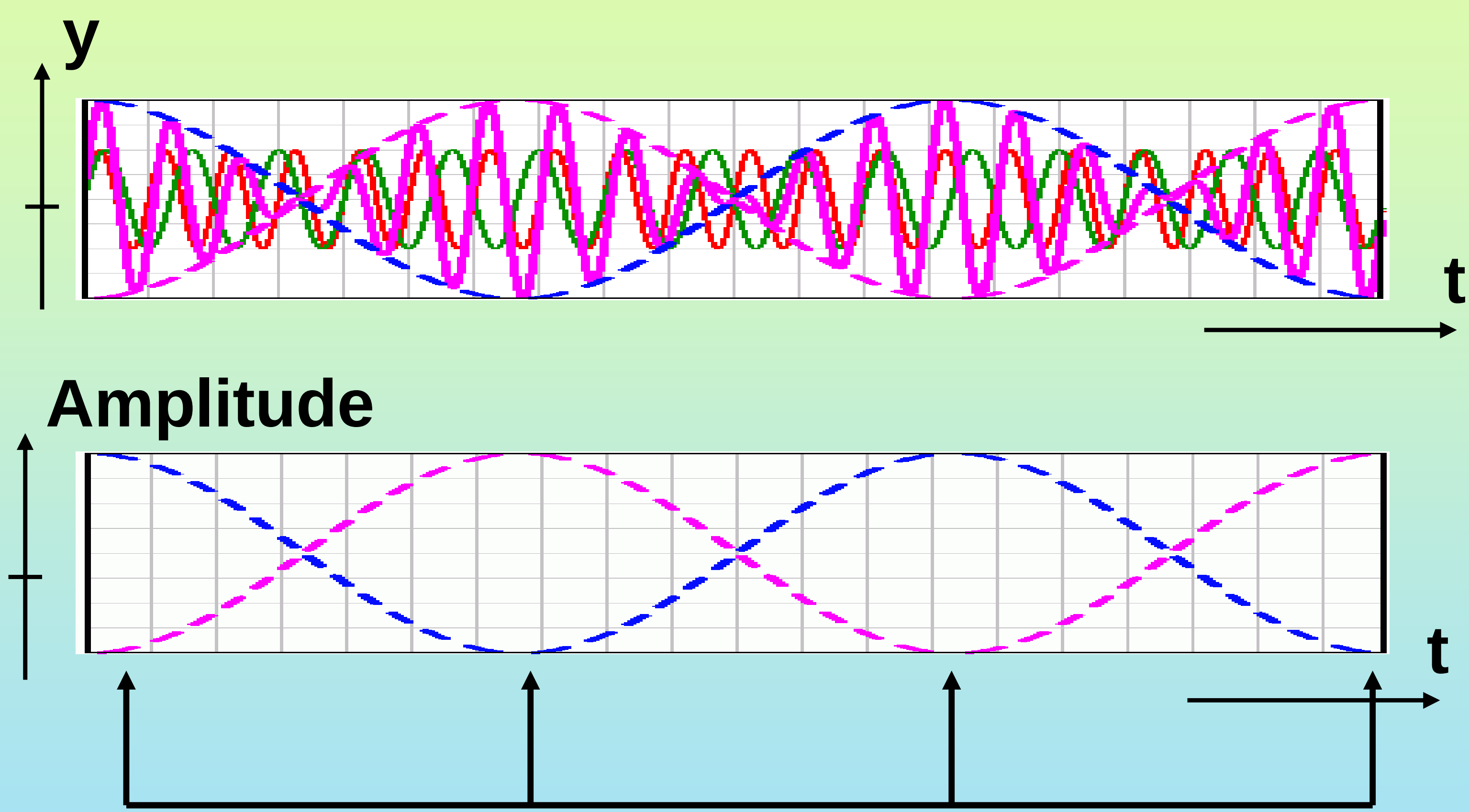
The sum wave = total.displacement



Envelope of sum wave showing the “beats”. A beat occurs at times of maximum sum wave amplitude.

The Envelope Wave:

The envelope of the sum wave represents its amplitude.



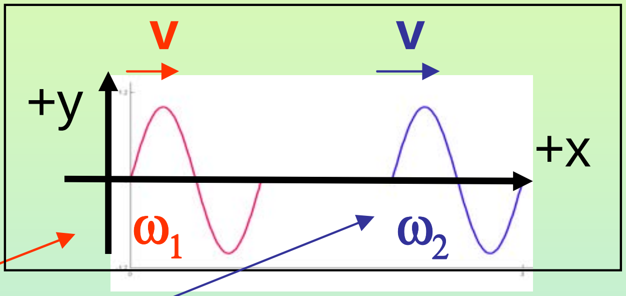
Beats occur where the amplitude is a maximum.

Beats Analytically :

Here we develop the wave function for beats.

Beats occur, when two waves of different frequencies traverse a medium in the same direction. We will assume harmonic waves along the x-axis.

We firstly write the wave function for each traveling wave



For both waves: a = amplitude

k = wave number

ω_1 and ω_2 = angular frequencies

ω_1 wave: $y_1 = a \sin(kx - \omega_1 t)$ ------(1)

ω_2 wave: $y_2 = a \sin(kx - \omega_2 t)$ ------(2)

where y_1 and y_2 are the displacements of the two waves

For convenience, let's choose the point P to be at the origin, where x = 0. Thus:

ω_1 wave: $y_1 = a \sin(\omega_1 t)$ ------(3)

ω_2 wave: $y_2 = a \sin(\omega_2 t)$ ------(4)

Beats Wave Function :

At Point P (x = 0), in the medium, at any time, t, the resultant displacement, y, will be:

$$
\begin{aligned}
y &= y_1 + y_2 \\
&= a\sin(\omega_1 t) + a\sin(\omega_2 t) \quad \{\text{from } (3) \,\&\, (4)\} \\
&= +2a\sin\left(\frac{\omega_1+\omega_2}{2}\right)t \quad \cos\left(\frac{\omega_1-\omega_2}{2}\right)t
\end{aligned}
$$

since: $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

Beat wave Function:

$$
\boxed{y = \left[2a\cos\left(\frac{\omega_1-\omega_2}{2}\right)t\right]\sin\left(\frac{\omega_1+\omega_2}{2}\right)t \quad \ldots\ldots\ldots(5)}
$$

Time-dependent amplitude term

The resultant wave at P consists of an harmonic wave $\boxed{\sin\left(\frac{\omega_1+\omega_2}{2}\right)t}$

whose angular frequency is the average of the two component waves, $\left(\frac{\omega_1+\omega_2}{2}\right)$

with a time-dependent amplitude $\boxed{\left[2a\cos\left(\frac{\omega_1-\omega_2}{2}\right)t\right]}$

of lower frequency, equal to half the difference between the components $\left(\frac{\omega_1-\omega_2}{2}\right)$

Beat Frequency:

$$y = \left[2a\cos\left(\frac{\omega_1 - \omega_2}{2}\right)t \right] \sin\left(\frac{\omega_1 + \omega_2}{2}\right)t \;\;.........(5)$$

We say that the resultant wave consists of a high-frequency carrier wave, with an amplitude that is modulated by a lower frequency wave.

Definition: The beat frequency, f_B, is the number of beats that occur per second.

From (5), the **angular frequency** of the amplitude term is $(\omega_1 - \omega_2)/2$..
The corresponding **frequency** will be $(f_1 - f_2)/2$

But a beat occurs when the amplitude is either +2a or -2a, that is, twice per cycle of the amplitude.

Thus the **beat frequency, f_B, will be $(f_1 - f_2)$**

The beat frequency is equal to the frequency difference between the two component waves

But we have assumed that $f_1 > f_2$. It could be that $f_1 < f_2$. Therefore:

$$f_B = | f_1 - f_2 |$$

Example:

When a tuning fork of frequency 440 Hz is sounded with a second tuning fork, 2 beats per second are heard. What is the frequency of the second fork?

Let f_{440} = frequency of the first fork
and f = frequency of the second fork

Beat frequency = $f_B = | f_{440} - f |$

Therefore $f = f_{440} +/- f_B$
$= 440 +/- 2 =$ $\boxed{442 \text{ Hz or } 438 \text{ Hz}}$

(There are **two** possible answers)

Beats In Music: When two notes are sounded together, the sound can seem pleasant (consonant), or harsh (dissonant). This is determined by the beat pattern that is produced. To hear some examples check the following url.

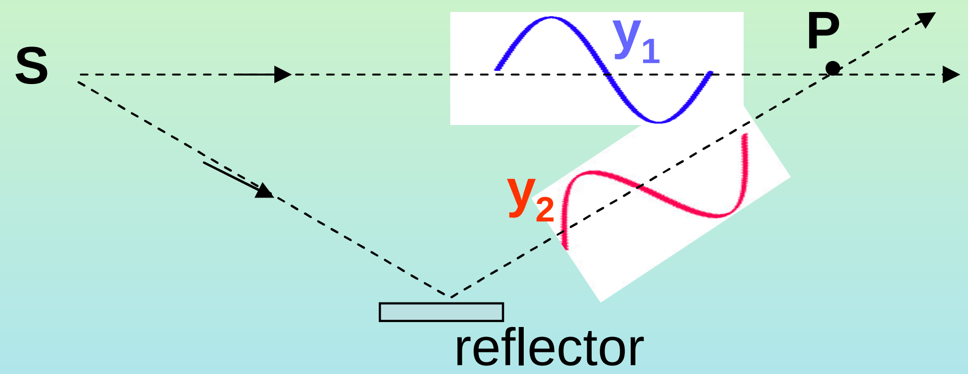
http://www.phys.unsw.edu.au/~jw/beats.html

3. Interference :

This is our third, and final, example of superposition.

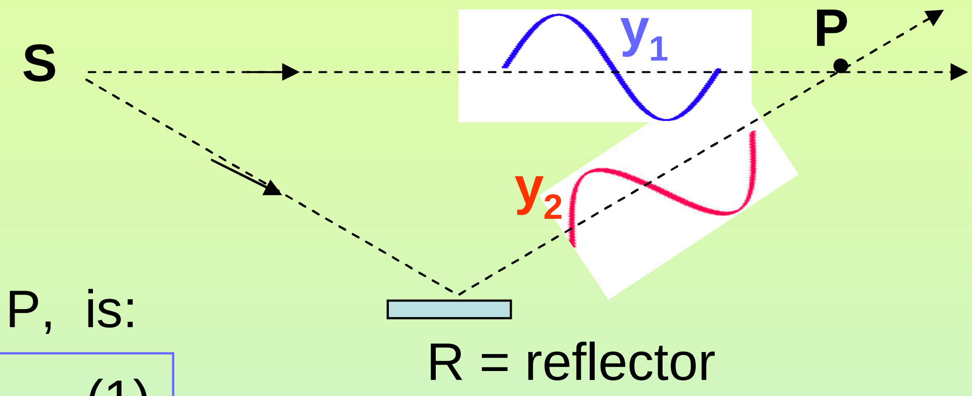
Interference occurs when two waves that differ only in phase, pass through the same point.

This difference in phase is typically due to a difference in path length traveled, from a common source, S, to some point P, where the interference happens.



If the two waves are in phase, at P, they will add together, producing a resultant wave of maximum amplitude; if they are 180 deg out of phase, they will cancel each other out, producing a null, or minimum. Thus, as we shift the point P, in space, we get a spatial pattern of maxima, and minima, called an interference pattern.

Interference Wave Function:



Say that wave 1 travels to point P by path SP, while wave 2 travels by longer path SRP.

Let the path difference = (SRP - SP) = Δ.

If SP = x, the wave function for wave 1, at P, is:

$$y_1 = a \sin(kx - \omega t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and for wave 2:

$$y_2 = a \sin(k[x + \Delta] - \omega t) \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Thus the resultant wave at P will be:

$$y = y_1 + y_2 = a\sin(kx - \omega t) + a\sin(k[x+\Delta] - \omega t)$$

$$= 2a\sin\left(\frac{2kx - 2\omega t + k\Delta}{2}\right)\cos\left(\frac{k\Delta}{2}\right)$$

$$y = \boxed{\left[2a\cos\left(\frac{k\Delta}{2}\right)\right]}\sin\left(kx - \omega t + \frac{k\Delta}{2}\right) \quad \ldots(3)$$

<u>since</u>: $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

amplitude term

$$y = \left[2a\cos\left(\frac{k\Delta}{2}\right) \right] \sin\left(kx - \omega t + \frac{k\Delta}{2} \right) \quad \ldots(3)$$

Intensity:

Interference can occur for waves of any type – for example, sound waves can interfere, as can radio waves, and light waves. In fact, interference is a fundamental, and characteristic property of waves in general.

In the case of light waves, the frequency is too high for current technology to directly observe wave displacements or amplitudes. Only wave **intensities** can be measured.

As we have seen earlier, intensity is proportional to the square of the amplitude of a wave. Thus the intensity of an interference pattern, I, can be got from the amplitude-

From (3), using $k \equiv \dfrac{2\pi}{\lambda}$:

$$\text{amplitude} = \left[2a\cos\left(\frac{k\Delta}{2}\right) \right] = \left[2a\cos\left(\pi\frac{\Delta}{\lambda}\right) \right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$\boldsymbol{I} \propto (\text{amplitude})^2 = 4a^2\cos^2\left(\pi\frac{\Delta}{\lambda}\right)$$

$$\therefore \boxed{I \propto 2a^2\left[\cos\left(2\boldsymbol{\pi}\frac{\Delta}{\lambda}\right) + 1\right]} \quad \ldots\ldots\ldots(5) \quad \left\{ \begin{array}{l} \text{using } \cos 2\theta = 2\cos^2\theta - 1 \\ \text{hence } \cos^2\theta = \frac{1}{2}(\cos 2\boldsymbol{\theta} + 1) \end{array} \right\}$$

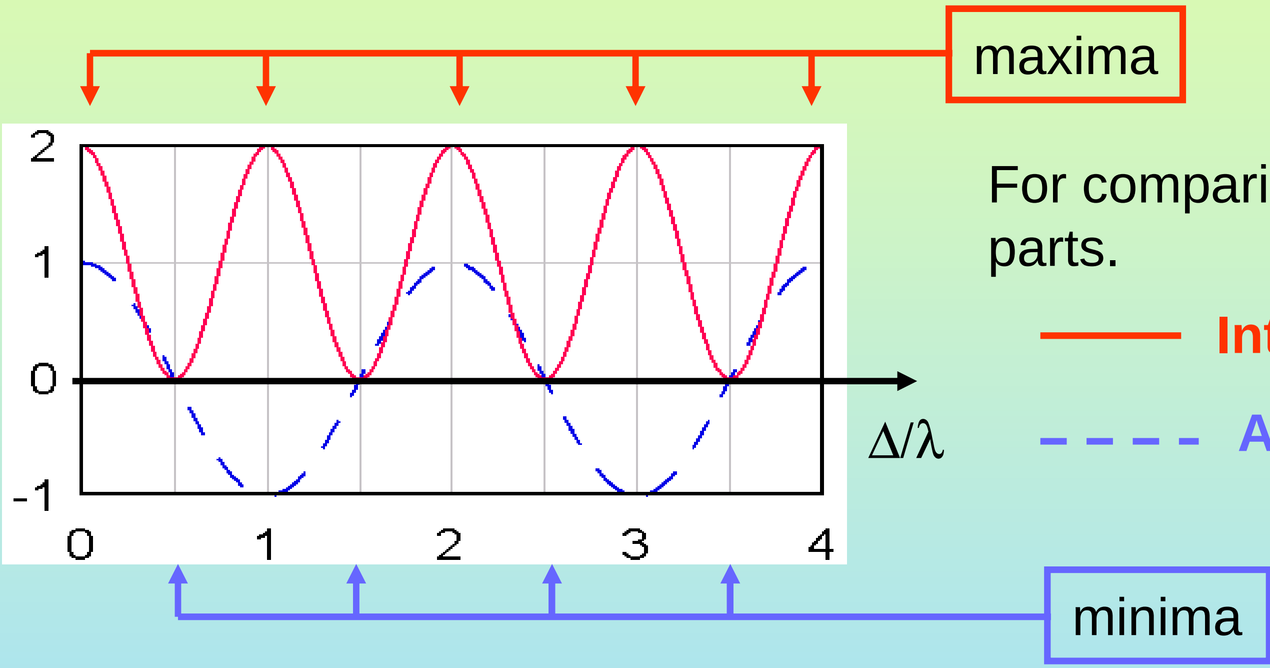
Intensity of the interference pattern

$$a = 2a\cos\left(\pi\frac{\Delta}{\lambda}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$I \propto 2a^2\left[\cos\left(2\pi\frac{\Delta}{\lambda}\right)+1\right] \quad \ldots\ldots\ldots(5)$$

Graphically:

We can plot the amplitude and intensity functions together, as the path difference increases.



For comparison, we only plot the cosine parts.

Intensity: $\cos(2\pi\,\Delta/\lambda) + 1$

Amplitude: $\cos(\pi\,\Delta/\lambda)$

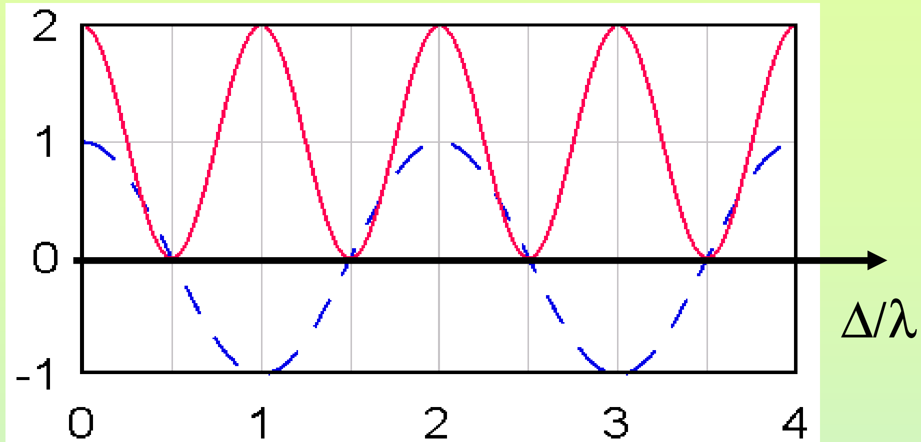
Note that the minimum intensity is zero – can’t be negative.

Note that the intensity is a maximum, where amplitude is (+2a) or (-2a).

Minima occur midway between maxima.

Maxima & Minima:

Maxima:
A maximum intensity occurs where
$(\Delta/\lambda) = 0, 1, 2, 3, ...$
$\Delta = 0, \lambda, 2\lambda, 3\lambda$ ....



A maximum occurs where there is a path difference equal to an **even number of wavelengths**

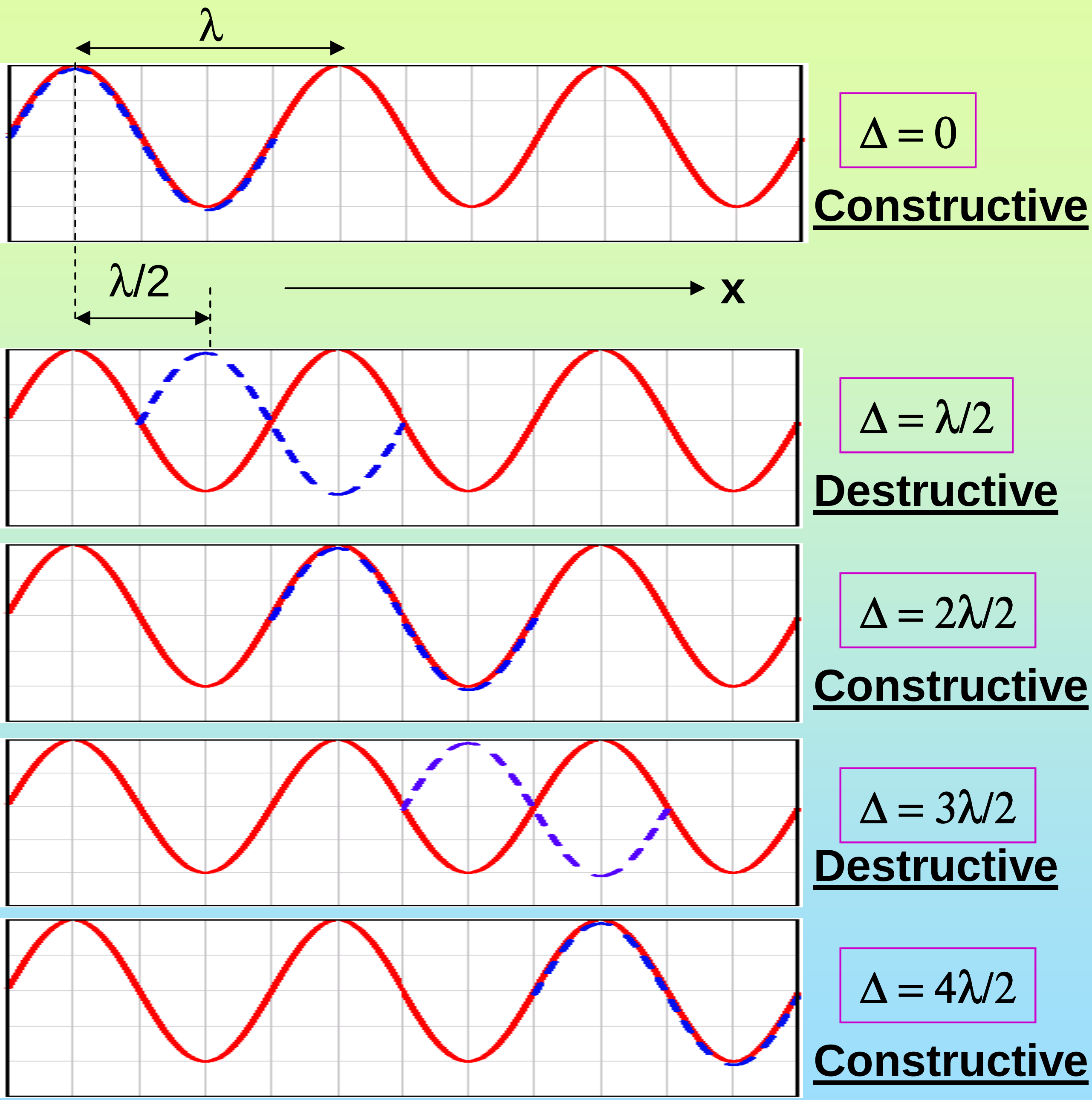
Minima:
A minimum amplitude occurs where
$(\Delta/\lambda) = 1/2, 3/2, 5/2, 7/2, ...$
$\Delta = \lambda/2, 3\lambda/2, 5\lambda/2, 7\lambda/2,$ ....

Increasing the path difference, Δ, between the two waves by $\lambda/2$, causes the sum wave at P, to go from a maximum to the next minimum,

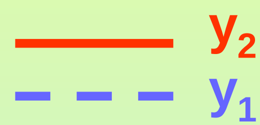
A minimum occurs where there is a path difference equal to an **odd number of half wavelengths**

Interference produces a pattern of maxima, and minima in space – the interference pattern.

In Terms of the Component Waves:



Consider the two interfering waves, at a given instant, at different points near P.



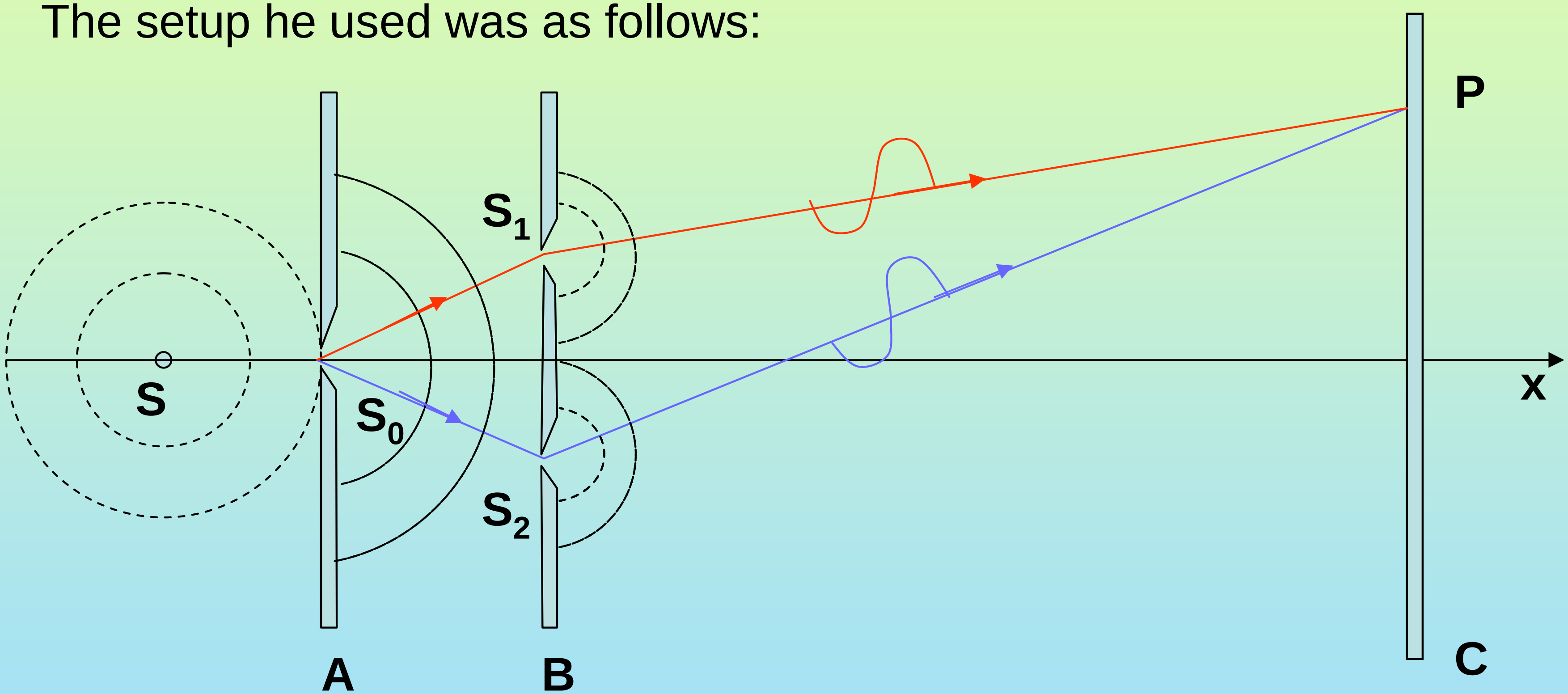
As the path difference, Δ, between the waves, increases in steps of $\lambda/2$, one wave shifts in phase, in steps of π, relative to the other, and the pattern at P goes from a maximum (constructive interference), to the next minimum (destructive interference).

Constructive interference is where the waves add to a maximum; **destructive interference** is where they add to zero.

Young's Double Slit Experiment:

Ever since the time of Newton, a controversy existed as to whether light was a stream of particles, or a wave motion. In 1801 a pivotal experiment was carried out by Thomas Young, which demonstrated conclusively that light behaved like a wave. He showed that light from two sources could produce interference effects.

The setup he used was as follows:



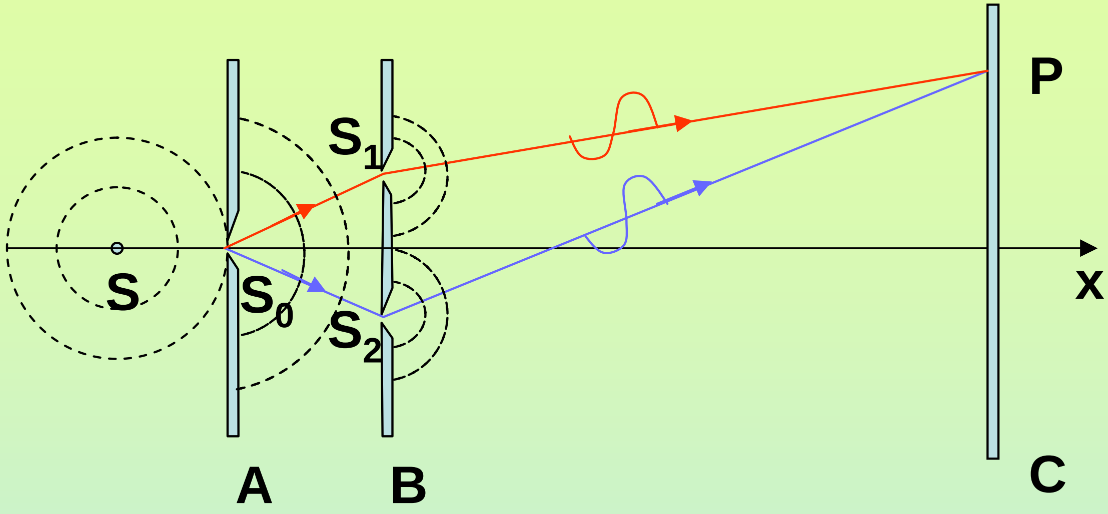
S = monochromatic (ie. single frequency) physical light source

A, B, & C are screens

S_0 = slit in screen A

S_1 & S_2 = pair of slits in screen B

The Experiment:



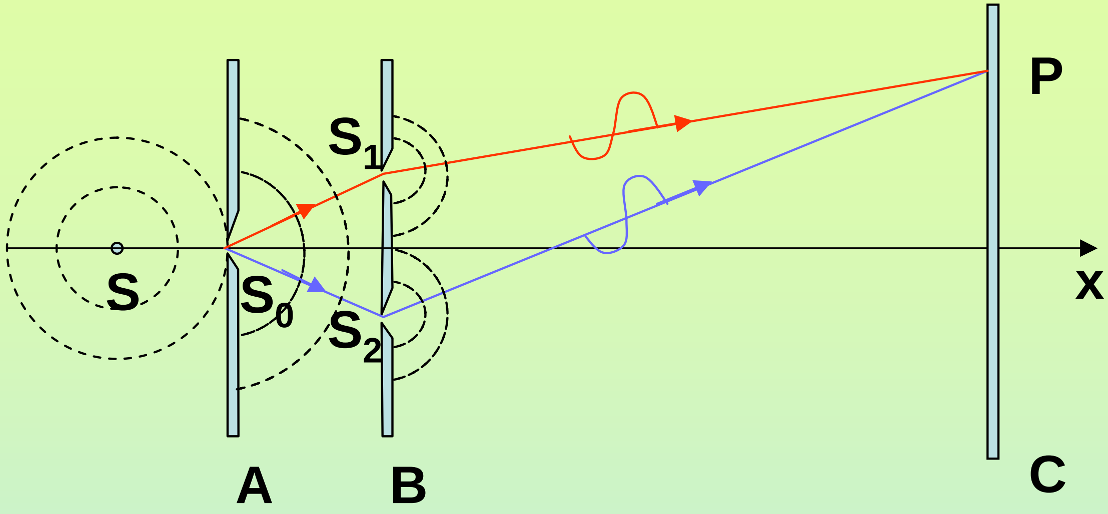
Wavefronts are emitted from physical light source S. These wavefronts, which may be irregular in shape, arrive at screen A. Slit S_0 selects a tiny sample of each of these total wavefronts, and transmits it on to screen B.

Because only a small section of the original irregular wavefront from S, was transmitted by S_0, the wavefronts arriving at S_1 & S_2 will be regular (smooth). At screen B, two sections of this regular wavefront are selected, simultaneously, by slits S_1 & S_2 and transmitted on to screen C. Thus S_1 & S_2, become two virtual sources of waves, that are in phase at S_1 & S_2, and are transmitted to screen C.

Consider a typical point, P, on screen C. Because the path lengths S_1P, and S_2P are generally different, the waves arriving at a particular point, P, will have a **fixed phase difference**. The particular phase difference will depend on where point P is located on the screen.

Thus we have the conditions for interference to occur at P – two waves that differ only in phase, passing through a given point in space.

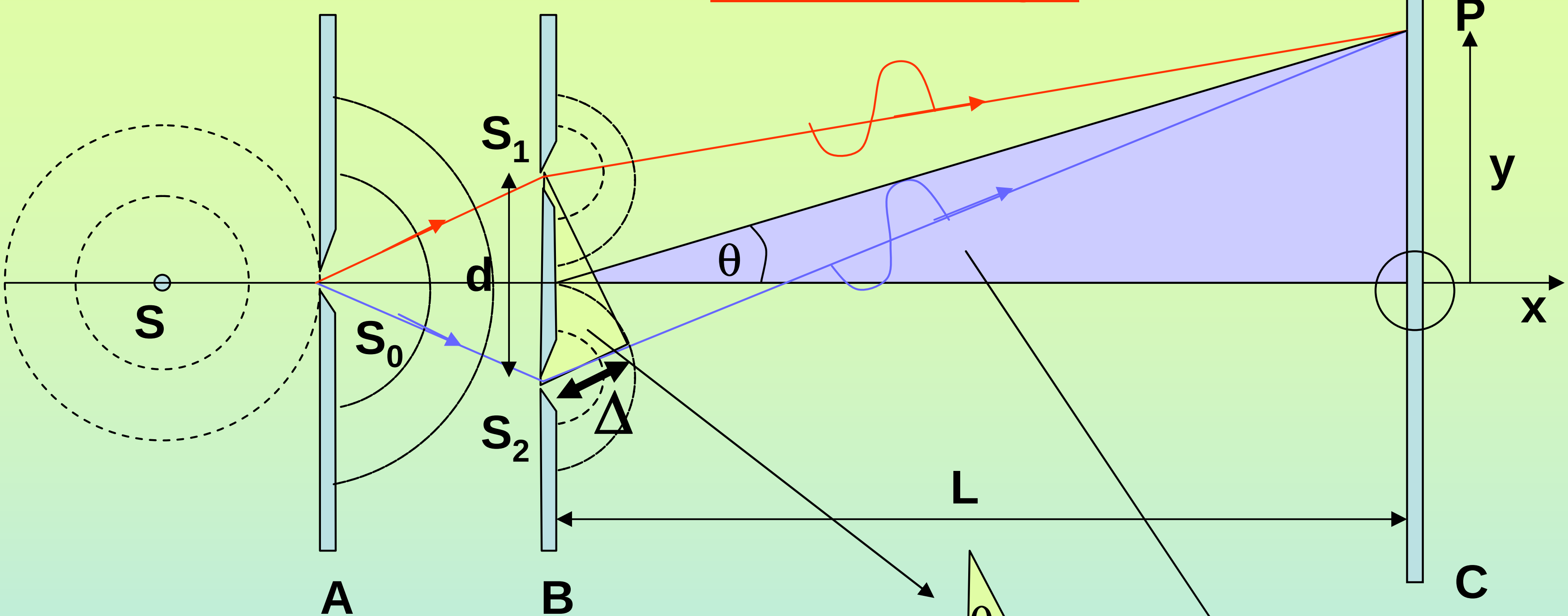
At Screen C:



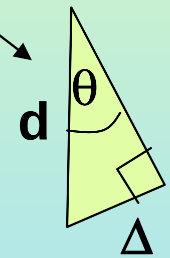
Optical interference will take place on screen C.
Where there is constructive interference between the two waves, we get a maximum light intensity (a bright fringe); where there is destructive interference we get a minimum light intensity (a dark fringe).

We will firstly, find an expression for the path difference between the two beams, from the geometry of the experiment.
We will then be able to find the positions for maxima, and minima.
Finally, we will find an expression for the intensity of the interference pattern.

Geometry:



Let x-axis be the symmetry axis passing mid-way between slits S_1 & S_2.



$\sin\theta = \Delta/d$
$\Delta = d \sin\theta$ ...(2)

Let:

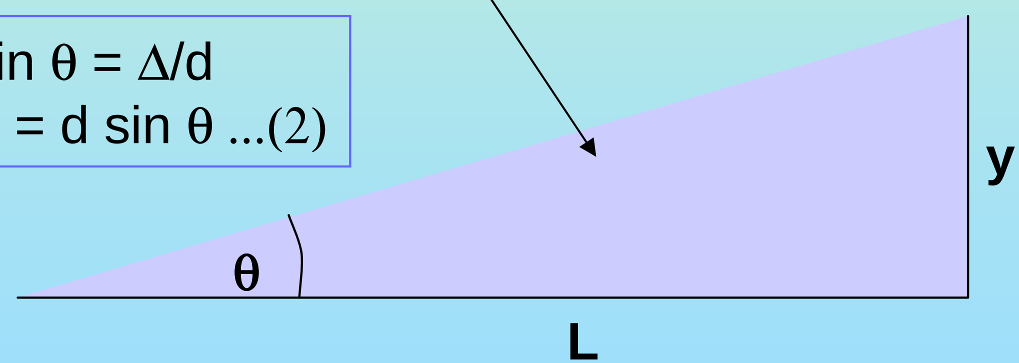
slit separation = S_1S_2 = d

distance from B to C = L

angle of P to x-axis = θ

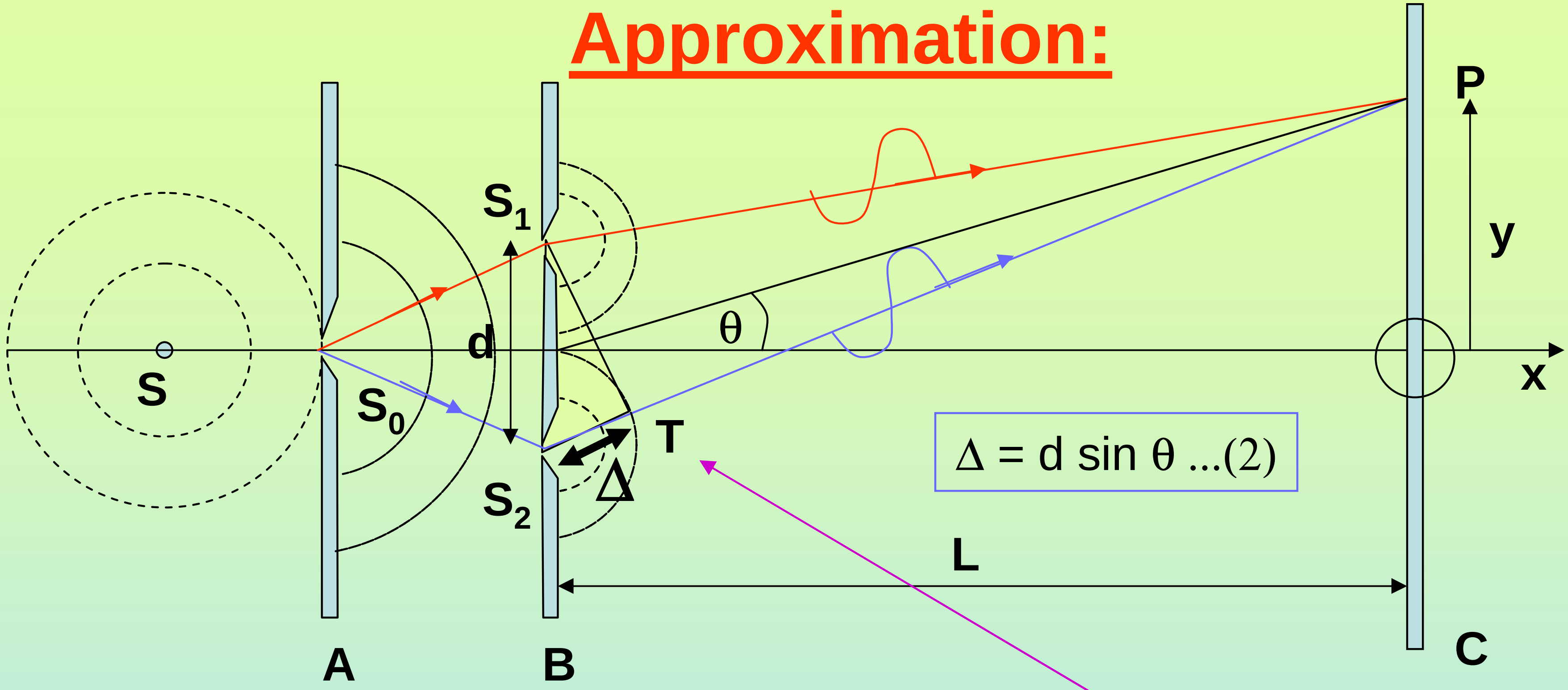
height of P above x-axis = y

path difference = $S_2P - S_1P = \Delta$



$\tan\theta = y/L$ ....(1)

Approximation:



In deriving equation (2), we have assumed that if S_1P = TP, then the angle at T is 90 deg.

This will only be precisely correct if S_1P is parallel to S_2P, which is clearly not so.

Therefore we must settle for the **assumption** that S_1P is approximately parallel to S_2P.

This approximation will be valid if L >>d. (or, that screen C is far from the slits, or that angle θ is small.)

(Conditions that satisfy this assumption are called **Fraunhoffer conditions.)**

$\tan\theta = y/L$ ....(1)

$\sin\theta = \Delta/d$

$\Delta = d \sin\theta$ ...(2)

Constructive Interference:

We saw earlier, for an **intensity maximum** (**constructive interference)**, that the path difference, Δ, needs to be a **whole number of wavelengths.**

$\Delta = 0, \lambda, 2\lambda, 3\lambda,$ ......

or $\Delta = d \sin\theta = m\lambda$ $(m = 0, 1, 2, 3, ...)$ from (1) **Max.**

Using (1), we can express this in terms of y:

$\Delta = d \sin\theta = d \tan\theta = (d/L)\, y = m\lambda$ $(m = 0, 1, 2, 3, ...)$ .....(3) **Max.**

$\sin\theta \sim \tan\theta$, since θ is small

Thus, from (3), y positions of intensity maxima are:

$y = m\,(L/d)\,\lambda$ $(m = 0, 1, 2, 3, ...)$ .....(4) **Max.**

Thus the intensity maxima occur as bright parallel bands of light ("bright fringes"), equally-spaced up the y-axis (parallel to the z-axis). From (4), the distance up the y-axis from one fringe to the next (the bright fringe spacing) = $(L/d)\lambda$)

$\tan\theta = y/L$ ....(1)

$\sin\theta = \Delta/d$
$\Delta = d\sin\theta$ ...(2)

Destructive Interference:

We also saw earlier, for an **intensity minimum** (**destructive interference)**, that the path difference, Δ, needs to be an **odd number of half wavelengths.**

$\Delta = \lambda/2, 3\lambda/2, 5\lambda/2, 7l/2,$ ......
$= (0+1/2)\lambda, (1+1/2)\lambda, (2+1/2)\lambda, (3+1/2)\lambda,$ .....
or $\Delta = d\sin\theta = (m+1/2)\,\lambda$ $(m = 0, 1, 2, 3, ...)$ from (1) **Min.**

Using (1), we can express this in terms of y:

$\Delta = d\sin\theta = d\tan\theta = (d/L)\,y = (m+1/2)\,\lambda$ $(m = 0, 1, 2, 3, ...)$ .....(3) **Min.**

$\sin\theta \sim \tan\theta$, since θ is small

Using (3), y positions of intensity maxima:

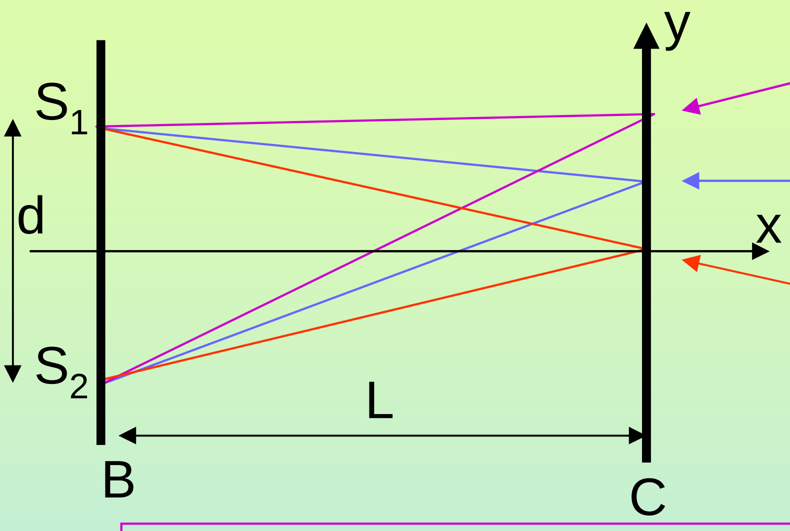
$y = (m+1/2)\,(L/d)\,\lambda$ $(m = 0, 1, 2, 3, ...)$ .....(5) **Min.**

Thus the intensity minima occur as dark parallel bands of light (“dark fringes”), equally-spaced up the y-axis, and midway between the bright fringes (From (4): dark fringe spacing = $(L/d)\lambda$)

Interference Pattern:

Consider screens B & C.

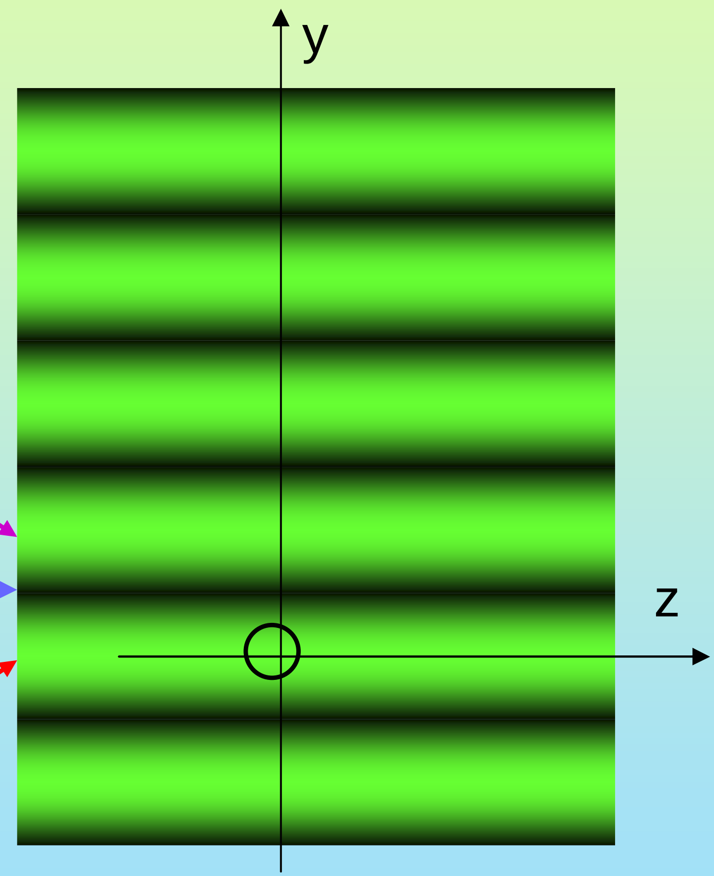
For m = 0, 1, 2, 3, ...
y = m (L/d) λ.....(4) **Max.**
y = (m+1/2) (L/d) λ ....(5) **Min.**



From (4):
m = 1 gives a **bright** fringe at $y = (L/d)\lambda$

From (5):
m = 0 gives a **dark** fringe at $y = (1/2)(L/d)\lambda$

From (4):
m = 0 gives a **bright** fringe at y = 0



The screen C will be crossed by alternating, equally spaced, bright and dark fringes aligned parallel to the z-axis.

Intensity Pattern:

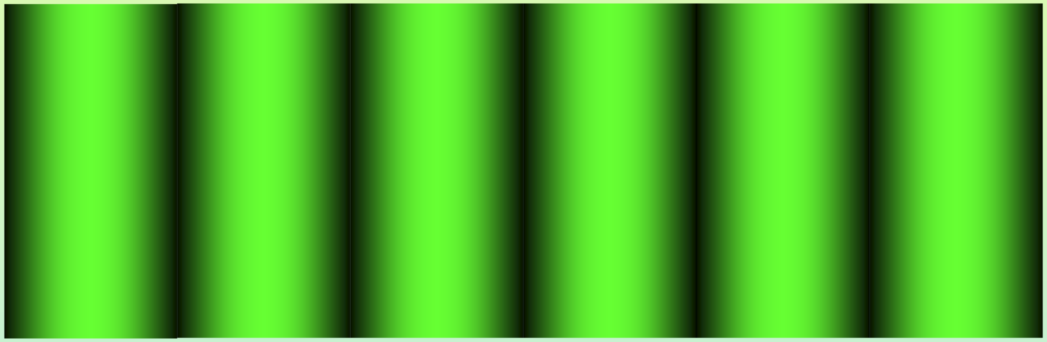
From our earlier general discussion of interference, we found that the intensity of an interference pattern is given by:

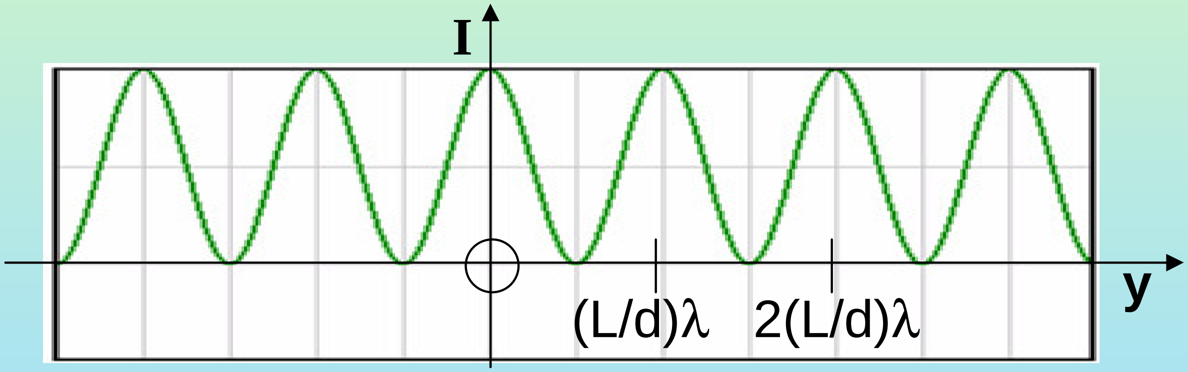
$$I \propto (\text{amplitude})^2 = 4a^2 \cos^2\left(\pi \frac{\Delta}{\lambda}\right)$$

$$\rightarrow \quad I = I_0 \cos^2\left(\pi \frac{\Delta}{\lambda}\right) \quad \ldots\ldots\ldots\ldots(6)$$

where I_0 is the maximum intensity (at the centre of a bright fringe)

And, here:
$\Delta = d \sin\theta = (d/L)y$





Hence, for the intensity pattern (we rotated it for convenience)

we can plot the intensity, I.

Example:

Two slits separated by 1.00 mm, are illuminated by a laser, so that an interference pattern is produced, on a screen, distant 2.00 m from the slits. The separation between adjacent bright fringes on the screen is measured to be 1.26 mm. What was the wavelength of the laser?

Fringe separation is given by: $\Delta y = (L/d)\,\lambda$

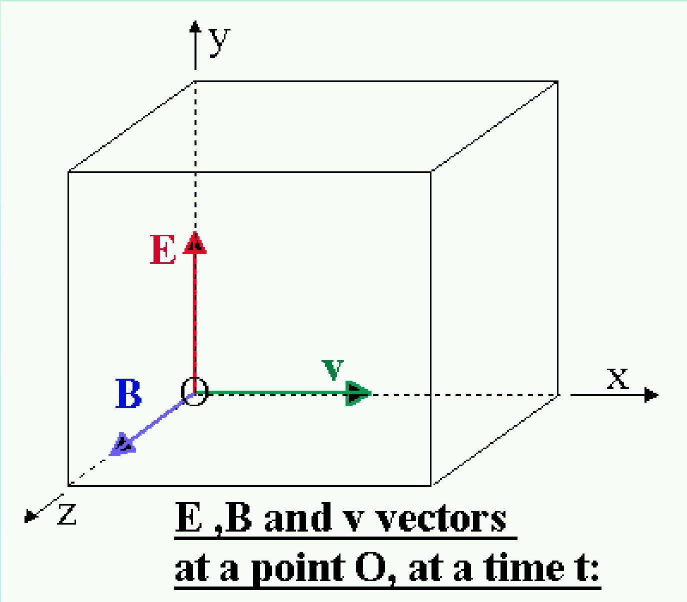
therefore

$$\lambda = (\Delta y)(d/L)$$
$$= (1.26\times10^{-3})\,(1.00\times10^{-3})\,/\,(2.00)$$
$$= 6.30\times10^{-7}\text{ m}$$
$$= 630\text{ nm}$$

The Nature of Light:

Young's experiment showed that light behaves like a **wave**. But what is it that vibrates, to produce the wave? – what sort of wave is light?

Light consists of a **vibrating electric field**. That is, at a point in space through which light passes, an electric field vector exists, which vibrates in length (that is, in strength), as the light wave propagates past the point. Associated with this electric field, there is a simultaneous **magnetic field** vector, at the same point, which is perpendicular to, and vibrates with, the electric vector. Both electric and magnetic vectors (**E** and **B**) are perpendicular to the direction of the wave velocity (**v**) of the light wave.



(The direction of the cross product of the field vectors, (**E**x**B**) , at any moment, is in the direction of the wave velocity, **v**.)

Because it is composed of vibrating electric, and magnetic, fields, we call light an **electromagnetic wave**. The energy of the wave is the total of the energies contained within the electric and magnetic fields.

Electromagnetic Waves:

For visible light (wavelength range about 400-750 nm), the frequency of the light wave is perceived, by the human eye, as colour.

Light of **lower frequency** appears **red**; light of **higher frequency** appears **blue**.

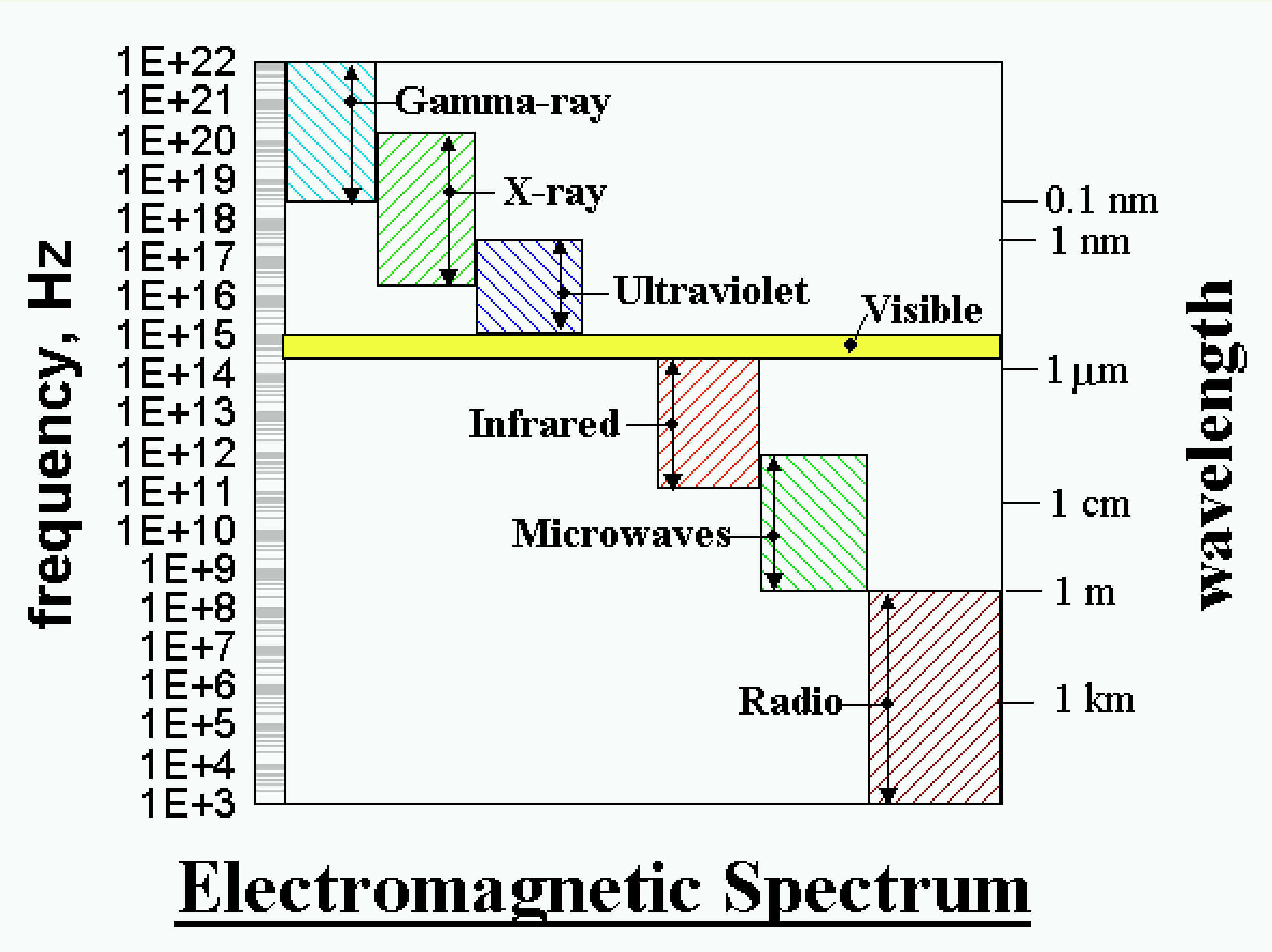
If we reduce the light frequency below red frequencies, or above blue frequencies, we get other types of electromagnetic wave, which are invisible to the human eye.

Lower frequencies: infra-red, microwave, radio waves.

Higher frequencies: ultra-violet, X-rays, γ-rays.

The frequencies of the Electromagnetic spectrum are illustrated in the following diagram:

EM Spectrum:



frequency, Hz
1E+22
1E+21
1E+20
1E+19
1E+18
1E+17
1E+16
1E+15
1E+14
1E+13
1E+12
1E+11
1E+10
1E+9
1E+8
1E+7
1E+6
1E+5
1E+4
1E+3
Gamma-ray
X-ray
Ultraviolet
Visible
Infrared
Microwaves
Radio
0.1 nm
1 nm
1 µm
1 cm
1 m
1 km
wavelength
Electromagnetic Spectrum

Waves or Particles?:

Young's experiment demonstrated, for the first time, that light is a **wave**.

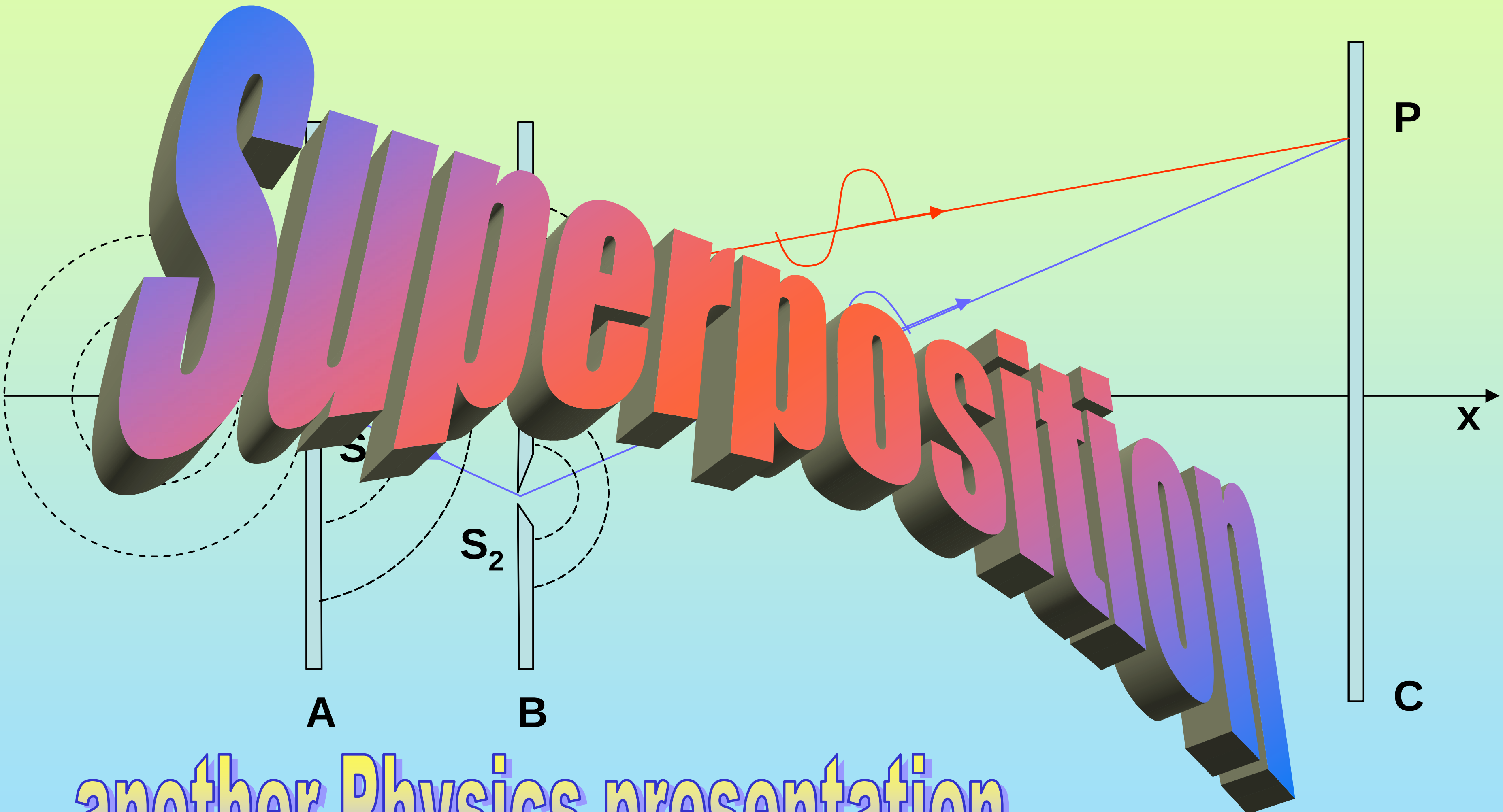
However, there are other experiments, such as the photoelectric effect , in which light behaves like a **stream of particles**, called photons. Photons are tiny discrete particles of electromagnetic energy.

The question therefore arises: is light a wave, or a stream of particles – which is the correct model?

It turns out that light, together with other forms of electromagnetic radiation, has a strange, dual nature – it shows **both wave and particle characteristics**.

Quantum Physics had to be developed to account properly for this dual character of light.

We hope you enjoyed -



another Physics presentation