UPC - 2351504 Prob & Stats V Sem FY UP

Sno. Q Paper 1357 held on 8/12/16. (I)

Q1. (i) A : chips have same no. $P(A) = {}^3C_1 / {}^8C_2$ — (1)

B : " " same colours $P(B) = ({}^5C_2 + {}^3C_2) / {}^8C_2$ — (1)

Reqd. Prob $= P(A \cup B)$ where $P(A \cap B) = 0$ as $A \cap B = \phi$

$= P(A) + P(B) = \dfrac{{}^3C_1 + {}^5C_2 + {}^3C_2}{{}^8C_2} = \dfrac{16}{28} = \dfrac{4}{7}$ — (1)

(ii) T.S. $b(x; n, \theta) = b(n-x, n, 1-\theta)$

$b(x; n, \theta) = {}^nC_x\, \theta^x (1-\theta)^{n-x}, \quad x = 0, 1, \ldots, n$

$= \dfrac{n!}{x!\,(n-x)!}\, \theta^x (1-\theta)^{n-x}$, "

$= \dfrac{n!}{(n-x)!\,(n-(n-x))!} (1-\theta)^{n-x} \cdot \theta^{(n-(n-x))}, \quad x = 0, 1, \ldots, n$ — (1)

$(\because x = n-(n-x))$

$= {}^nC_{n-x} (1-\theta)^{n-x}\, \theta^{n-(n-x)}, \quad n-x = n, n-1, \ldots, 1, 0$ — (1)

$= {}^nC_y (1-\theta)^y\, \theta^{n-y}, \quad y = 0, 1, \ldots, n; \ (y = n-x)$

$= b(y; n, (1-\theta))$ — (1)

$= b(n-x; n, (1-\theta))$

(iii) $f_X(x) = \begin{cases} cx^3, & 0 < x < 2 \\ 0 & \text{else} \end{cases}$

By property of pdf $\int_0^2 f_X(x)\,dx = 1$ i.e. $\dfrac{c\,x^4}{4}\Big|_0^2 = 1$ i.e. $4c = 1$ or $c = 1/4$ — (1)

$\therefore P\left(\frac{1}{4} < X < 1\right) = \int_{1/4}^{1} f_X(x)\,dx = \frac{1}{4}\left(\frac{x^4}{4}\Big|_{1/4}^{1}\right) = \frac{255}{4096} = 0.06225$ — (2)

(iv) To find $E(X_2 | x_1)$ at $x_1 = 1$. $p_{X_2|x_1}(x_2) = \begin{cases} 4/7, & x_2 = 0 \\ 3/7, & x_2 = 1 \\ 0, & \text{else} \end{cases}$ — (1)

$X_1 \backslash X_2$	0	1	$p_{X_1}(x_1)$
0	1/18	3/18	4/18
1	4/18	3/18	7/18
2	6/18	1/18	7/18
$p_{X_2}(x_2)$	11/18	7/18	1

(1)

So $E(X_2|x_1) = 0 \cdot \frac{4}{7} + 1 \cdot \frac{3}{7} = \frac{3}{7}$ — (1)

where $\because p_{X_2|x_1}(x_2) = \dfrac{p_{X_1,X_2}(x_1, x_2)}{p_{X_1}(x_1)}$

(v) $E(X-b)^2 = E(X^2 - 2bX + b^2) = E(X^2) + b^2 - 2bE(X)$

$$= (\sigma^2+\mu^2) + b^2 - 2b\mu \quad \text{where } \mu = E(X) \;\&\; \sigma^2 = \mathrm{var}(X)$$

$$= f(b), \text{ say } \quad -①$$

To minimise f, $f'(b) = 0$ (necessary condition)

i.e. $2(b-\mu) = 0$

i.e. $b = \mu$ $\quad -①$

Also $f''(b) = 2$

$\therefore f''(\mu) = 2 > 0$ $\quad -①$ $\therefore$ by II^nd derivative test $b=\mu$ is a pt. of minima for f.

(vi) For x_0 to be median

$$P(X < x_0) = P(X \le x_0) = \frac{1}{2} \quad -①$$

$\therefore \int_{-\infty}^{x_0} \frac{1}{\pi(1+x^2)}\,dx = \frac{1}{2}$ gives $\frac{1}{\pi}\left[\tan^{-1}x_0 + \frac{\pi}{2}\right] = \frac{1}{2}$

or $\boxed{x_0 = \tan^{-1}0 = 0}$ $\quad -②$

(vii) $\mu = 500$ $\quad \sigma^2 = 100 \neq 0$ $\therefore \sigma = 10$

By Chebychev's Inequality: $P(|X-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}$ $\quad -①$

so, $P(|X-500| < 100) = P(|X-\mu| < k\sigma)$ where $k = 10$

$$\geq 1 - \frac{1}{k^2}$$

$$= 1 - \frac{1}{10^2} = \frac{99}{100} = 0.99 \quad ②$$

Q2.(i) Put $\theta = \lambda/n$ in $b(x;n,\theta)$

$$b(x;n,\theta) = {}^nC_x\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x}, \quad x = 0,1,\ldots,n$$

$$= \frac{n(n-1)\cdots(n-x+1)}{x!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x} \quad -②$$

Dividing one of the x factors n in $\left(\frac{\lambda}{n}\right)^x$ into each factor of the product $n(n-1)\ldots(n-x+1)$ & writing $\left(1-\frac{\lambda}{n}\right)^{n-x}$ as $\left(\left(1-\frac{\lambda}{n}\right)^{-\frac{n}{\lambda}}\right)^{-\lambda}\left(1-\frac{\lambda}{n}\right)^{-x}$

we get $1.\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)\lambda^x\left[\left[1-\frac{\lambda}{n}\right]^{-\frac{n}{\lambda}}\right]^{-\lambda}\left(1-\frac{\lambda}{n}\right)^{-x}$ $\quad -②$

Letting $n\to\infty$ & $x;\lambda$ are held fixed

$1.\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{x-1}{n}\right) \longrightarrow 1$

$\left(1-\frac{\lambda}{n}\right)^{-x} \longrightarrow 1$

$\left(1-\frac{\lambda}{n}\right)^{-\frac{n}{\lambda}} \longrightarrow e$

$-①$

$\therefore$ limiting distribution is

$$p(x;\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0,1,2,\ldots$$

$-①$

(c) See Thm 6.8 Freund Page 216

$M_Z(t) = M_{\frac{X-\mu}{\sigma}}(t) = e^{-\mu t/\sigma} \cdot \left[1+\theta(e^{t/\sigma}-1)\right]^n$, $\mu = n\theta$, $\sigma = \sqrt{n\theta(1-\theta)}$

Take log & substitute Mclaurin's Series of $e^{t/\sigma}$

$$\ln\left(M_{\frac{X-\mu}{\sigma}}(t)\right) = -\frac{\mu t}{\sigma} + n \cdot \ln\left[1+\theta(e^{t/\sigma}-1)\right]$$

$$= -\frac{\mu t}{\sigma} + n\ln\left[1+\theta\left[\frac{t}{\sigma}+\frac{1}{2}\left(\frac{t}{\sigma}\right)^2+\frac{1}{6}\left(\frac{t}{\sigma}\right)^3+\cdots\right]\right]$$

& using inf. series $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$, which converges for $|x|<1$, to expand this logarithm

$$\ln M_{\frac{X-\mu}{\sigma}}(t) = -\frac{\mu t}{\sigma} + n\theta\left[\frac{t}{\sigma}+\frac{1}{2}\left(\frac{t}{\sigma}\right)^2+\cdots\right] - \frac{n\theta^2}{2}\left[\frac{t}{\sigma}+\frac{1}{2}\left(\frac{t}{\sigma}\right)^2+\cdots\right]^2 + \frac{n\theta^3}{3}\left(\frac{t}{\sigma}+\frac{1}{2}\left(\frac{t}{\sigma}\right)^2+\cdots\right)^3 - \cdots$$

collecting powers of t:

$$= \left(-\frac{\mu}{\sigma}+\frac{n\theta}{\sigma}\right)t + \left(\frac{n\theta}{2\sigma^2}-\frac{n\theta^2}{2\sigma^2}\right)t^2 + \left(\frac{n\theta}{6\sigma^3}-\frac{n\theta^2}{2\sigma^3}+\frac{n\theta^3}{3\sigma^3}\right)t^3+\cdots$$

$$= \frac{1}{\sigma^2}\left(\frac{n\theta-n\theta^2}{2}\right)t^2 + \frac{n}{\sigma^3}\left(\frac{\theta-3\theta^2+2\theta^3}{6}\right)t^3 + \cdots$$

$\because \mu = n\theta$. Putting $\sigma = \sqrt{n\theta(1-\theta)}$:

$$\ln\left[M_{\frac{X-\mu}{\sigma}}(t)\right] = \frac{1}{2}t^2 + \frac{n}{\sigma^3}\left(\frac{\theta-3\theta^2+2\theta^3}{6}\right)t^3 + \cdots$$

for $r>2$ the coeff of t^r is constt. times $\frac{n}{\sigma^r}$ which tends to 0 as $n\to\infty$

i.e. $\lim_{n\to\infty} \ln M_{\frac{X-\mu}{\sigma}}(t) = \frac{t^2}{2}$.

Take exp. on both sides ($\because$ log is cont)

we get $\lim_{n\to\infty} M_{\frac{X-\mu}{\sigma}}(t) = e^{t^2/2} = M_Z(t)$ where $Z \sim N(0,1)$

Hence Proved

2(iii) Let X have geom. distn $g(x;\theta) = \theta(1-\theta)^{x-1}$, $x = 1, 2, 3, \ldots$

$$M_X(t) = E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx} g(x;\theta)$$

$$= \sum_{x=1}^{\infty} e^{tx}\theta(1-\theta)^{x-1}$$

$$= \theta\left[\sum_{x=1}^{\infty} e^{tx}(1-\theta)^{x-1}\right]$$ geom series c.r. $= e^t(1-\theta)$

$= \theta e^t / (1-e^t(1-\theta))$

(2)

$$\mu_1' = \frac{d}{dt} M_x(t) \Big|_{t=0} = \frac{\theta e^t}{(1 - e^t(1-\theta))^2} \Big|_{t=0} = \frac{1}{\theta} = \mu \quad -(2)$$

$$\sigma^2 = \mu_2' - \mu_1'^2 = \frac{2-\theta}{\theta^2} - \frac{1}{\theta^2} = \frac{1-\theta}{\theta^2}$$

$$\because \mu_2' = \frac{d^2}{dt^2} M_x(t) \Big|_{t=0} = \frac{(1-e^t(1-\theta))^2 \theta e^t - \theta e^t [2(1-e^t(1-\theta))(-(1-\theta)e^t)}{(1-e^t(1-\theta))^4}$$

$$= \frac{\theta^3 - 2\theta^2(\theta-1)}{\theta^4} = \frac{2-\theta}{\theta^2} \quad -(2)$$

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(5)

(i) Gamma fn: $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy \quad \forall \alpha > 0$ & $\alpha > 1$

$\Rightarrow \Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$

$\Gamma(1) = 1$,

$\Gamma(1/2) = \sqrt{\pi}$.

Gamma distn

$$g(x;\alpha,\beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} \quad \forall x > 0 \;\&\; \alpha > 0,\ \beta > 0.$$

Thm 6.2 Pg 205 Freund

Thm 6.3

By defn,

$$\mu_r' = \int_0^\infty x^r \cdot \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} dx = \frac{\beta^r}{\Gamma(\alpha)} \int_0^\infty y^{\alpha+r-1} e^{-y} dy$$

Putting $y = x/\beta$

$$\because \int_0^\infty y^{\alpha+r-1} e^{-y} dy = \Gamma(r+\alpha)$$

So, we get $\mu_r' = \dfrac{\beta^r \Gamma(\alpha+r)}{\Gamma(\alpha)}$ — (3)

Putting $r=1$ & $r=2$,

$$\mu_1' = \frac{\beta\Gamma(\alpha+1)}{\Gamma(\alpha)} = \alpha\beta$$

$$\& \quad \mu_2' = \frac{\beta^2 \Gamma(\alpha+2)}{\Gamma(\alpha)} = \alpha(\alpha+1)\beta^2$$

(3)

So, $\mu = \alpha\beta$ & $\sigma^2 = \alpha(\alpha+1)\beta^2 - (\alpha\beta)^2 = \alpha\beta^2$

(ii) Let Joint pdf of X_1 & X_2 be

$$f(x_1,x_2) = \begin{cases} 8x_1x_2 & 0 < x_1 < x_2 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

eg 2.1.5 Pg 94 H&L

To find $E(7X_1X_2^2 + 5X_2)$

$$E(X_1X_2^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2^2 f(x_1,x_2)\,dx_1\,dx_2$$

$$= \int_0^1 \int_0^{x_2} 8x_1^2 x_2^3\, dx_1\, dx_2$$

$$= \int_0^1 \frac{8}{3} x_2^6\, dx_2 = \frac{8}{21}$$ — (2)

$$E(X_2) = \int_0^1 \left(\int_0^{x_2} x_2\, 8(x_1x_2)\, dx_1 \right) dx_2$$

$$= \int_0^1 4x_2^4\, dx_2 = \frac{4}{5}$$ (2)

Obs. $f_2(x_2) = \begin{cases} 4x_2^3, & 0 < x_2 < 1 \\ 0, & \text{else} \end{cases}$

$\therefore E(X_2) = \int_0^1 x_2 4x_2^3\, dx_2 = \frac{4}{5}$

$\therefore E(7X_1X_2^2 + 5X_2)$

By linearity of E

$= 7E(X_1X_2^2) + 5E(X_2)$

$= 7 \times \frac{8}{21} + 5 \times \frac{4}{5} = \frac{20}{3}$

— (2)

Q3(iii) Given: and $f_{1|2}(x_1|x_2) = c_1 x_1/x_2^2,\ 0<x_1<x_2, \ldots$ (6)

$f_2(x_2) = c_2 x_2^4,\ 0<x<1$, zero else

a) by property of conditional pdf $\int_{-\infty}^{\infty} f_{1|2}\, dx_1 = 1$

$\therefore \int_0^{x_2} f_{1|2}(x_1|x_2)\,dx_1 = 1$ i.e. $\left.\frac{c_1 x_1^2}{2x_2^2}\right|_0^{x_2} = 1 \quad \therefore \frac{c_1}{2} = 1 \therefore c_1 = 2$ — (1)

IIIly $\int_0^1 f_2(x_2)\,dx_2 = 1$ i.e. $\left.\frac{c_2 x_2^5}{5}\right|_0^1 = 1$ i.e. $c_2 = 5$ — (1)

(b) Let $0<x_1<x_2<1$

Then joint pdf $f_{12}(x_1,x_2) = f_2(x_2)\cdot f_{1|2}(x_1|x_2) = 5x_2^4 \cdot \frac{2x_1}{x_2^2}$

$= \begin{cases} 10x_1x_2^2, & 0<x_1<x_2<1 \\ 0 & \text{else} \end{cases}$

— (1)

(c) $P\left(\frac{1}{4}<X_1<\frac{1}{2} \,\middle|\, X_2 = 5/8\right) = \int_{1/4}^{1/2} f_{1|2}\left(x_1 \,\middle|\, \frac{5}{8}\right) dx_1$

$= \int_{1/4}^{1/2} \frac{2x_1}{25} \times 64\, dx_1 = \frac{128}{50}\left[x_1^2\right]_{1/4}^{1/2} = \frac{64}{25}\left[\frac{1}{4}-\frac{1}{16}\right] = \frac{12}{25}$

— (1)

(d). $P\left(\frac{1}{4}<X_1<\frac{1}{2}\right) = \int_{1/4}^{1/2} f_1(x_1)\,dx_1$

$f_1(x_1) = \int_{x_1}^{1} f(x_1,x_2)\,dx_2 = 10x_1\left[\frac{x_2^3}{3}\right]_{x_1}^{1} = \frac{10x_1}{3} - \frac{10x_1^4}{3}$, $0<x_1<1$, 0 else — (1)

so, $P\left(\frac{1}{4}<X_1<\frac{1}{2}\right) = \int_{1/4}^{1/2} \frac{10}{3}\left(x_1 - x_1^4\right)dx_1 = \frac{449}{3\times 512}$ — (1)

Q4(i) a) $P\left(X_1 \le \frac{1}{2}\right) = \int_0^{1/2}\int_0^1 f(x,y)\,dy\,dx = \int_0^{1/2} f_1(x)\,dx = \frac{3}{8}$ — (1)

$\because f_1(x) = \int_0^1 (x+y)\,dy = \begin{cases} x+\frac{1}{2}, & 0<x<1 \\ 0 & \text{else} \end{cases}$ — (1)

b) $P(X_1+X_2 \le 1) = \int_0^1\int_0^{1-x} (x+y)\,dy\,dx = \int_0^1 \left(x(1-x) + \frac{(1-x)^2}{2}\right)dx$ — (1)

$= \int_0^1 \left(\frac{1}{2} - \frac{1}{2}x^2\right)dx = \frac{1}{3}$

— (1)

(i) $\rho_{XY} = \frac{cov(X,Y)}{\sigma_1 \sigma_2} = \frac{E(XY) - \mu_1 \mu_2}{\sigma_1 \sigma_2}$

$\mu_1 = E(X) = \int_0^1 \int_0^1 x(x+y)\,dx\,dy = 7/12$

$\sigma_1^2 = E(X^2) - \mu_1^2 = \int_0^1 \int_0^1 x^2(x+y)\,dx\,dy - \left(\frac{7}{12}\right)^2 = \frac{11}{144}$ – ①

by symmetry $\mu_2 = E(Y) = \frac{7}{12}$, $\sigma_2^2 = \frac{11}{144}$

$cov(X,Y) = \int_0^1 \int_0^1 xy(x+y)\,dx\,dy - \left(\frac{7}{12}\right)^2 = -\frac{1}{144}$ – ①

$\therefore \rho_{XY} = -\frac{1}{11}$

Q4(ii) a) $E(X_1 | X_2 = x_2)$

Marginal pdf of X_2 is $f_2(x_2) = \int_0^{x_2} 21 x_1^2 x_2^3\,dx_1 = \begin{cases} 7x_2^6, & 0 < x_2 < 1 \\ 0 & \text{else} \end{cases}$

& $f_{1|2}(x_1|x_2) = \frac{21 x_1^2 x_2^3}{7 x_2^6} = \frac{3x_1^2}{x_2^3}$; $0 < x_1 < x_2$ — ①

zero else

$\therefore E(X_1|x_2) = \int_0^{x_2} x_1 \frac{3x_1^2}{x_2^3}\,dx_1 = \frac{3}{x_2^3}\left(\frac{x_1^4}{4}\Big|_0^{x_2}\right) = \frac{3}{x_2^3}\left[\frac{x_2^4}{4}\right]$

$= \frac{3x_2}{4}$, $0 < x_2 < 1$ – ①

$Var(X_1|x_2) = \int_0^{x_2} \left(x_1 - \frac{3x_2}{4}\right)^2 \frac{3x_1^2}{x_2^3}\,dx_1$ – ①

$\int_0^{x_2} (x_1 - \mu_{X_1|x_2})^2 f_{1|2}(x_1|x_2)\,dx_1$

$= \frac{3}{80} x_2^2$. – ①

Q4(ii) $Y = E(X_1|X_2) = \frac{3X_2}{4}$, $0 < x_2 < 1$

cdf of Y $G(y) = P(Y \le y) = P\left(X_2 \le \frac{4y}{3}\right)$, $0 \le y < \frac{3}{4}$

from pdf of $f_2(x_2)$ we have

$G(y) = \int_0^{4y/3} 7x_2^6\,dx_2 = \frac{7x_2^7}{7}\Big|_0^{\frac{4y}{3}} = \left(\frac{4}{3}\right)^7 y^7$, $0 \le y < \frac{3}{4}$ – ①

pdf is $g(y) = \frac{d}{dy} G(y) = \begin{cases} 7\left(\frac{4}{3}\right)^7 y^6, & 0 < y < \frac{3}{4} \\ \text{zero else} \end{cases}$ – ①

Q4.(iii) a) $M(t_1,t_2) = \int_0^{\infty}\int_x^{\infty} e^{(t_1x+t_2y-y)}\,dy\,dx$ — ①

$= \dfrac{1}{(1-t_1-t_2)(1-t_2)}$ if $t_1+t_2<1$ & $t_2<1$ — ①

b) $\text{cov}(X,Y) = E(XY) - \mu_1\mu_2 = E(X-\mu_1)(Y-\mu_2)$.

As, $\mu_1 = E(X) = \dfrac{\partial}{\partial t_1}M(0,0) = \dfrac{1}{(1-t_2)}\left(\dfrac{1}{(1-t_1-t_2)^2}\right)\Bigg|_{(0,0)} = 1$

— ①

$\mu_2 = E(Y) = \dfrac{\partial}{\partial t_2}M(0,0) = \dfrac{1}{(1-t_2)(1-t_1-t_2)^2} + \dfrac{1}{(1-t_2)^2(1-t_1-t_2)}\Bigg|_{(0,0)}$

$= 1+1 = 2$

$E(X-\mu_1)(Y-\mu_2) = \underbrace{\dfrac{\partial^2}{\partial t_1\partial t_2}M(0,0)}_{E(XY)} - \mu_1\mu_2 = 3 - 2\times 1$

$= 1.$ — ②

(or) $\sigma_1^2 = E(X^2) - \mu_1^2 = \dfrac{\partial^2 M(0,0)}{\partial t_1^2} - \mu_1^2 = 1$.

$\sigma_2^2 = E(Y^2) - \mu_2^2 = \dfrac{\partial^2 M(0,0)}{\partial t_2^2} - \mu_2^2 = 2$

(c) $\because M(t_1,t_2) \neq M(t_1,0)M(0,t_2) \therefore$ r.v's are dependent

— ①

Q5(i) writing $f(y|x) = \dfrac{f(x,y)}{g(x)}$

let $u = \dfrac{x-\mu_1}{\sigma_1}$ $v = \dfrac{y-\mu_2}{\sigma_2}$

$$f(y|x) = \dfrac{\dfrac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\, e^{-\frac{1}{2(1-\rho^2)}[u^2 - 2\rho uv + v^2]}}{\dfrac{1}{\sqrt{2\pi}\sigma_1}\, e^{-\frac{1}{2}u^2}}$$

$$= \dfrac{1}{\sqrt{2\pi}\sigma_2\sqrt{1-\rho^2}}\, e^{-\frac{1}{2(1-\rho^2)}[v^2 - 2\rho uv + u^2\rho^2]}$$

$$= \dfrac{1}{\sqrt{2\pi}\sigma_2\sqrt{1-\rho^2}}\, e^{-\frac{1}{2}\left[\frac{v-\rho u}{\sqrt{1-\rho^2}}\right]^2}$$

$$= \frac{1}{\sigma_2 \sqrt{2\pi}\sqrt{1-\rho^2}} e^{-\frac{1}{2}\left[\frac{y - \{\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)\}}{\sigma_2\sqrt{1-\rho^2}}\right]} \quad \text{for } -\infty < y < \infty$$ (9)

Comparing this with Normal pdf

$$\text{Mean } \mu_{Y|x} = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)$$

$$\& \text{ Var } \sigma_{Y|x}^2 = \sigma_2^2(1-\rho^2).$$

Q5(ii) Integrating out y we find that marginal density of X is given by $g(x) = \begin{cases} e^{-x}, & x>0 \\ 0, & \text{else} \end{cases}$

$$g(x) = \int_0^\infty x e^{-x(1+y)} dy = -e^{-x(1+y)}\Big|_0^\infty = e^{-x} \quad -(2)$$

$\therefore$ conditional density of Y given $X=x$

$$f(y|x) = \frac{f(x,y)}{g(x)} = x e^{-xy} \text{ for } y>0, \text{ zero else} \quad -(2)$$

This is Exponential Density with $\theta = \frac{1}{x}$

$X \sim \text{exp}(\theta)$ $g(x;\theta) = \begin{cases} \frac{1}{\theta} e^{-x/\theta}, & x>0 \\ 0, & \text{else} \end{cases}$ where $\theta>0$

$$\therefore \mu_{Y/x} = \int_0^\infty y \cdot x \, e^{-xy} dy = \frac{1}{x} \quad -(2)$$

(iii) $\mu_{Y|x} = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)$ Ans: $= \frac{2}{3}(1-x)$

$$f_1(x) \overset{0<x<1}{=} \int_0^{1-x} 24xy\,dy = \begin{cases} 12x(1-x)^2 \\ \text{else} \end{cases}$$

$$\mu_1 = E(X) = \int_0^1 12x^2(1+x^2-2x)dx = \frac{2}{5} \quad -(1)$$

$$E(X^2) = \int_0^1 12x^3(1+x^2-2x)dx = \frac{1}{5} \quad -(1)$$

$$\sigma_1^2 = \frac{1}{5} - \frac{4}{25} = \frac{1}{25} \quad -(1)$$

By symmetry, $\mu_2 = \frac{2}{5}$, $E(Y^2) = \frac{1}{5}$ & $\sigma_2^2 = \frac{1}{25}$.

$$E(XY) = \int_0^1\int_0^{1-x} x^2y^2 \cdot 4\,dy\,dx = \int_0^1 8x^2(1-x)^3dx = 8\times\frac{1}{60} = \frac{2}{15} \quad -(1)$$

so, $\rho = \dfrac{E(XY) - \mu_1\mu_2}{\sigma_1\sigma_2}$

$= \dfrac{\frac{2}{15} - \frac{4}{25}}{\frac{1}{25}} = \dfrac{10-12}{\cancel{25}\times 3 \cdot \frac{1}{\cancel{25}}} = -\dfrac{2}{3}$ — (1)

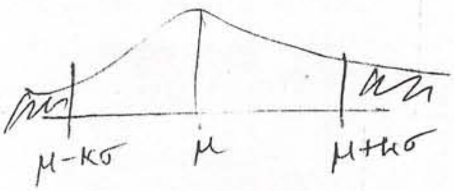
$\therefore \mu_{Y/x} = \mu_2 + \rho\dfrac{\sigma_2}{\sigma_1}(x-\mu_1) = \dfrac{2}{5} - \dfrac{2}{3}\left(x - \dfrac{2}{5}\right)$

$= \dfrac{2}{3}(1-x)$ — (1)

Statement — (2)

Q6: (i) If μ & σ are the mean & s.d. of r.v. X then for any +ive constt k: $P(|X-\mu| < k\sigma) \geq 1 - \dfrac{1}{k^2}$, $\sigma \neq 0$

$\sigma^2 = E(X-\mu)^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$



$= \underbrace{\int_{-\infty}^{\mu-k\sigma}}_{\mathrm{I}} + \underbrace{\int_{\mu-k\sigma}^{\mu+k\sigma}}_{\mathrm{II}} + \underbrace{\int_{\mu+k\sigma}^{\infty}}_{\mathrm{III}}$

$\geq \mathrm{I} + \mathrm{III} \quad (\because (x-\mu)^2 f(x) \geq 0)$

(2) — $\geq \int_{-\infty}^{\mu-k\sigma} k^2\sigma^2 f(x)\,dx + \int_{\mu+k\sigma}^{\infty} k^2\sigma^2 f(x)\,dx$ $\left(\because x \leq \mu - k\sigma \text{ or } x \geq \mu + k\sigma\right)$

$\therefore \dfrac{1}{k^2} \geq \underbrace{\int_{-\infty}^{\mu-k\sigma} f(x)\,dx + \int_{\mu+k\sigma}^{\infty} f(x)\,dx}$ if $\sigma^2 \neq 0$

$= P(X \leq \mu - k\sigma) + P(X \geq \mu + k\sigma)$

(2) — So $P(|X-\mu| \geq k\sigma) \leq \dfrac{1}{k^2}$

$\therefore P(|X-\mu| < k\sigma) \geq 1 - \dfrac{1}{k^2}$

(a) $\mu = n\theta = 20$ $\sigma^2 = n\theta(1-\theta) = 10$

$\theta = 1/2$

Binomial is discrete & normal is cont. distn

$$\therefore P(X=20) = P(19.5 < X < 20.5)$$

$$= P\left(\frac{19.5-20}{\sqrt{10}} < \frac{X-20}{\sqrt{10}} < \frac{20.5-20}{\sqrt{10}}\right)$$

$$= P\left(-0.16 < \frac{X-20}{\sqrt{10}} < 0.16\right)$$

$$\simeq \Phi(0.16) - \Phi(-0.16)$$

$$\Phi(x) = P(Z \leq x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\,dy$$

$$\Phi(-0.16) = P(Z > 0.16) = 1 - \Phi(0.16)$$

$$\therefore P(X=20) \simeq 2\Phi(0.16) - 1 = 2 \times 0.5636 - 1 = 0.1272$$

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Exact result is $^{40}C_{20} \left(\frac{1}{2}\right)^{40} = 0.1268$ — 1

(iii) State 0 : when it rains
State 1 : when it does not rain
This is a 2 state Markov chain with
Transition Prob Matrix $P = \begin{Vmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{Vmatrix}$ — 2

$\alpha = 0.7$
$\beta = 0.4$

So $P^2 = \begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix}$ — 2

$P^4 = P^2 \cdot P^2 = \begin{pmatrix} 0.5749 & 0.4251 \\ 0.5668 & 0.4332 \end{pmatrix}$ — 2

reqd Prob $= P^4_{00} = 0.5749$

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