Sequence of functions

- **Definition:**

  A sequence of functions is simply a set of functions $u_n(x)$, $n = 1, 2, . . .$ defined on a common domain D.

- A frequently used example will be the sequence of functions $\{1, x, x^2, . . .\}$, $x \in [-1, 1]$

Sequence of Functions Convergence

- Let D be a subset of $\Re$ and let $\{u_n\}$ be a sequence of real valued functions defined on D. Then $\{u_n\}$ converges on D to g if

  $$\lim_{n\to\infty} u_n(x) = g(x)$$

  for each x ϵ D
- More formally, we write that

  $$\lim_{n\to\infty} u_n = g$$

  if given any x ϵ D and given any ε > 0, there exists a natural number N = N(x, ε) such that

  $$\left|u_n(x) - g(x)\right| < \varepsilon, \quad \forall n \geq N$$

Sequence of Functions Convergence

- **Example 1**

  Let $\{u_n\}$ be the sequence of functions on $\Re$ defined by $u_n(x) = nx$.

  This sequence does not converge on $\Re$ because $\lim_{n\to\infty} u_n(x) = \infty$ for any $x > 0$

Sequence of Functions Convergence

- **Example 2**: Consider the sequence of functions

  $$u_n(x) = \frac{1}{1+nx}, \quad |x| < \infty, \quad n = 1, 2, 3, \ldots.$$

  The limits depends on the value of *x*

  We consider two cases, $x = 0$ and $x \neq 0$

  1. x = 0 → $\lim_{n\to\infty} u_n(0) = \lim_{n\to\infty} 1 = 1$

  2. x ≠ 0 → $\lim_{n\to\infty} u_n(x) = \lim_{n\to\infty} \frac{1}{1+nx} = 0$

Sequence of Functions Convergence

Therefore, we can say that $\{u_n\}$ converges to g for $|x| < \infty$, where

$$g(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Sequence of Functions Convergence

- **Example 3**:

  Consider the sequence $\{u_n\}$ of functions defined by

  $$u_n(x) = \frac{nx + x^2}{n^2}, \quad \text{for all } x \text{ in } \Re$$

  Show that $\{u_n\}$ converges for all x in $\Re$

Sequence of Functions Convergence

- Solution

  For every real number x, we have

$$\lim_{n\to\infty} u_n\left(x\right) = \lim_{n\to\infty}\frac{x}{n} + \frac{x^2}{n^2} = x\left(\lim_{n\to\infty}\frac{1}{n}\right) + x^2\left(\lim_{n\to\infty}\frac{1}{n^2}\right) = 0 + 0 = 0$$

  Thus, $\{u_n\}$ converges to the zero function on $\Re$

Sequence of Functions Convergence

- **Example 4**:

  Consider the sequence $\{u_n\}$ of functions defined by

  $$u_n(x) = \frac{\sin(nx+3)}{\sqrt{n+1}}, \quad \text{for all } x \text{ in } \Re$$

  Show that $\{u_n\}$ converges for all x in $\Re$

Sequence of Functions Convergence

- Solution

  For every real number x, we have

$$\frac{-1}{\sqrt{n+1}} \leq \frac{\sin(nx+3)}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n+1}}$$

Moreover,

$$\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$$

Applying the squeeze theorem, we obtain that

$$\lim_{n\to\infty} u_n(x) = 0, \quad \text{for all } x \text{ in } \Re$$

Therefore, $\{u_n\}$ converges to the zero function on $\Re$

Sequence of Functions Convergence

- **Example 6**

  Consider the sequence $\{f_n\}$ of functions defined by $f_n(x) = x^n, \quad x \in [0,1], \quad n = 1, 2, \ldots.$

  We recall that the definition for convergence suggests that for each x we seek an N such that $\left| f_n(x) - g(x) \right| < \varepsilon, \quad \forall n \geq N$.

  This is not at first easy to see.

  So, we will provide some simple examples showing how N can depend on both x and ε

Sequence of Functions Convergence

1. $x = 0$. *Here we have* $f_n(0) = 0$ *for all* n. *So, given* $\epsilon > 0$ *we seek an* N *such that* $|f_n(0) - 0| < \epsilon, \forall n \geq N$. *Inserting* $f_n(0) = 0$, *we have* $0 < \epsilon$. *Since this is true for all* n, *we can pick* $N = 1$.

2. $x = \frac{1}{2}$. *In this case we have* $f_n(\frac{1}{2}) = \frac{1}{2^n}$, *for* $n = 1, 2, \ldots$. *As* n *gets large,* $f_n \to 0$. *So, given* $\epsilon > 0$, *we seek* N *such that* $|\frac{1}{2^n} - 0| < \epsilon, \quad \forall n \geq N$. *This means that* $\frac{1}{2^n} < \epsilon$. *Solving the inequality for* n, *we have* $n > -\frac{\ln \epsilon}{\ln 2}$ *We choose* $N \geq -\frac{\ln \epsilon}{\ln 2}$. *Thus, our choice of* N *depends on* ϵ. *For,* $\epsilon = 0.1$, *this gives*

$$N \geq -\frac{\ln 0.1}{\ln 2} = \frac{\ln 10}{\ln 2} \approx 3.32.$$

So, we pick $N = 4$ *and we have* $n > N = 4$.

Sequence of Functions Convergence

3. $x = \frac{1}{10}$. *This can be examined like the last example. We have* $f_n(\frac{1}{10}) = \frac{1}{10^n}$, *for* $n = 1, 2, \ldots$. *This leads to* $N \geq -\frac{\ln \epsilon}{\ln 10}$ *For* $\epsilon = 0.1$, *this gives* $N \geq 1$, *or* $n > 1$.

4. $x = \frac{9}{10}$. *This can be examined like the last two examples. We have* $f_n(\frac{9}{10}) = \left(\frac{9}{10}\right)^n$, *for* $n = 1, 2, \ldots$. *So given an* $\epsilon > 0$, *we seek an* N *such that* $\left(\frac{9}{10}\right)^n < \epsilon$ *for all* $n > N$. *Therefore,*

$$n > N \geq \frac{\ln \epsilon}{\ln\left(\frac{9}{10}\right)}.$$

For $\epsilon = 0.1$, *we have* $N \geq 21.85$, *or* $n > N = 22$.

So, for these cases, we have shown that N *can depend on both* x *and* ϵ.

Series of Functions

- **Definition:**

  An infinite series of functions is given by

  $\sum_{n=1}^{\infty} u_n(x),$ x ∈ D.

Series of Functions Convergence

- $\sum u_j(x)$ is said to be convergent on D if the sequence of partial sums $\{S_n(x)\}$, n = 1, 2, ..., where $S_N(x) = \sum_{n=1}^{N} f_n(x)$ is convergent on D
- In such case we write $\lim_{n\to\infty} S_n(x) = S(x)$ and call S(x) the sum of the series
- More formally,

  if given any x ϵ D and given any ε > 0, there exists a natural number N = N(x, ε) such that

$$\left| S_n(x) - S(x) \right| < \varepsilon, \quad \forall n \geq N$$

Series of Functions Convergence

- If N depends only on ε and not on x, the series is called uniformly convergent on D.

Series of Functions Convergence

- **Example 8**:

  Find the domain of convergence of $(1 – x) + x(1 – x) + x^2(1 – x) + ...$

Exercise

1. Consider the sequence $\{f_n\}$ of functions defined by $f_n(x) = n^2 x^n$ for $0 \le x \le 1$. Determine whether $\{f_n\}$ is convergent.
2. Let $\{f_n\}$ be the sequence of functions defined by $f_n(x) = \cos^n(x)$ for $-\pi/2 \le x \le \pi/2$. Determine the convergence of the sequence.
3. Consider the sequence $\{f_n\}$ of functions defined by $f_n(x) = nx(1-x)^n$ on $[0, 1]$

   Show that $\{f_n\}$ converges to the zero function

Exercise

4. Find the domain of convergence of the series

a) $\sum_{n=1}^{\infty} \frac{x^n}{n^3}$ b) $\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{2^n (3n-1)}$

c) $\sum_{n=1}^{\infty} \frac{1}{n\left(1+x^2\right)^n}$ d) $\sum_{n=1}^{\infty} n^2 \left(\frac{1-x}{1+x}\right)^n$

e) $\sum_{n=1}^{\infty} \frac{e^{nx}}{n^2-n+1}$

5. Prove that $\sum_{n=1}^{\infty} \frac{1.3.5...(2n-1)}{2.4.6...(2n)} x^n$ converges for $-1 \leq x < 1$

Exercise

6. Investigate the uniform convergence of the series $\sum_{n=1}^{\infty} \frac{x}{[1+(n-1)x][1+nx]}$

7. Let $f_n(x) = \frac{1}{1+nx}, \quad 0 < x < 1, \quad n = 1, 2, 3, \ldots$

   Prove that $\{f_n\}$ converges but not uniformly on $(0, 1)$