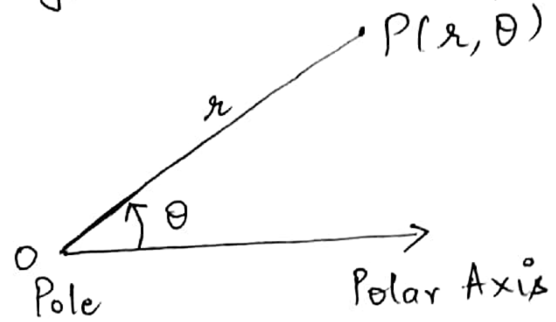
Polar Coordinates

The polar coordinate system uses the polar coordinates to specify the location of a point in the plane. Unlike, cartesian coordinates, a point in the plane has infinitely many pairs of polar coordinates.

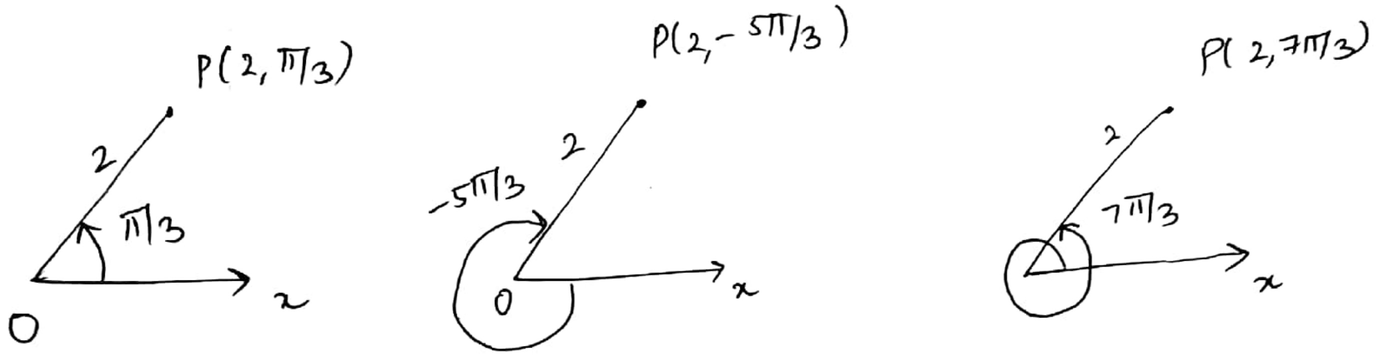
To define polar coordinates, we fix a point O, called the pole (or origin), an construct from O an initial ray, called the polar axis.



Then each point P in the plane can be assigned polar coordinates (r, θ) where r is the directed distance from O to P and θ is the directed angle from the polar axis to the ray $\overline{OP}$.

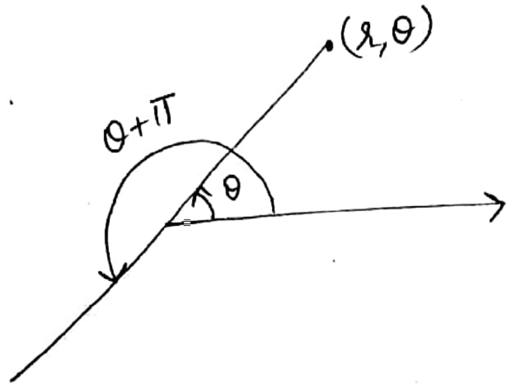
We use the convention that θ is positive when measured anticlockwise and negative when measured clockwise. We call r the radial coordinate and θ the angular coordinate (or polar angle) of P. The origin O has no well-defined angular coordinates, so we assign to O the polar coordinate $(0, \theta)$ for any angle θ.

<u>Note</u>; The angular coordinate associated with a given point is not unique. For example, the polar coordinates $\left(2, \frac{\pi}{3}\right)$, $\left(2, -\frac{5\pi}{3}\right)$ and $\left(2, \frac{7\pi}{3}\right)$ all represent the same point P



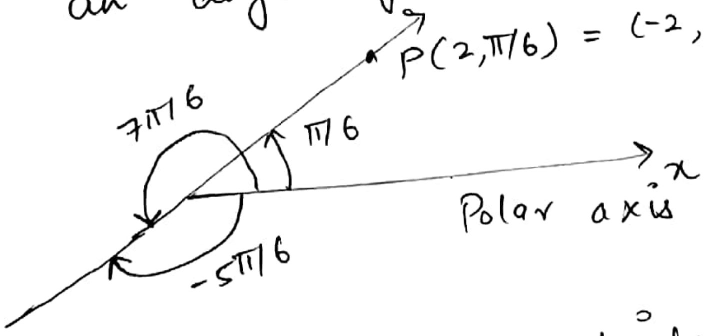
<u>Note</u>: (1) Because changing the angle by 2π does not change the point, the coordinates (r, θ) and $(r, \theta + 2n\pi)$ represent the same point for any integer n.

(2) Because r is a directed distance, we allow r to be negative with the convention that the points $(-r, \theta)$ and (r, θ) lie on the same line through the origin O and at the same distance $|r|$ from O. If $r > 0$, the point lies in the same quadrant as θ. If $r < 0$, it lies in the quadrant on the opposite side of the pole. With this convention, (r, θ) and $(-r, \theta + \pi)$ represent the same point. Thus, the point (r, θ) can be written as $(r, \theta) = (r, \theta + 2n\pi)$ or $(r, \theta) = (-r, \theta + (2n+1)\pi)$ where n is any integer.



Q Plot the points P whose polar coordinates are $(2, \pi/6)$. Find all the polar coordinates of P.

Solution: By fixing the pole O, we first draw the polar axis and then draw the ray from O that makes an angle of $\pi/6$ with polar axis.



Note that P has polar coordinate $(2, \pi/6)$ or $(-2, 7\pi/6)$ or $(-2, -5\pi/6)$. Infact, the point P $(2, \pi/6)$ has infinitely many polar coordinates which correspond to $r = 2$ and $r = -2$, respectively.

For $r = 2$, the complete list of polar coordinates is

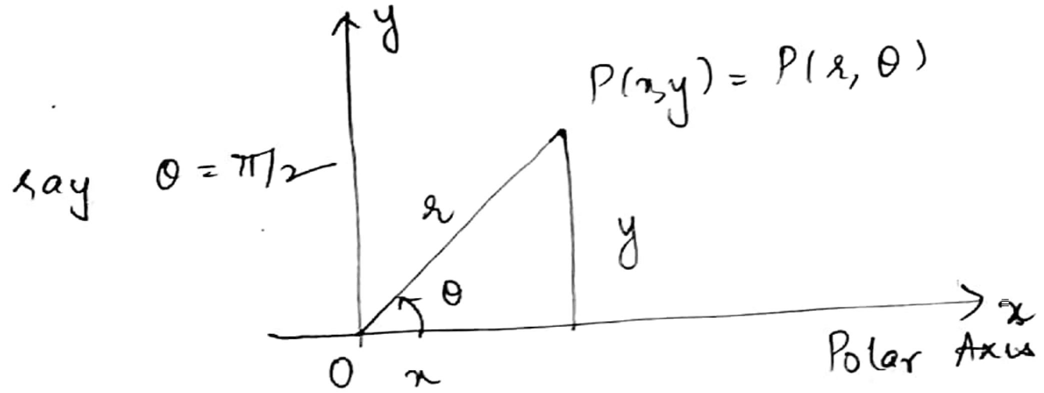
$$\left\{ \left(2, \frac{\pi}{6} + 2n\pi\right) ; n \text{ is an integer} \right\}$$

For $r = -2$, the complete list of polar coordinates is

$$\left\{ \left(-2, \frac{\pi}{6} + (2n+1)\pi\right) ; n \text{ is an integer} \right\}$$

Conversion between polar and cartesian coordinates

To establish the conversion between polar and cartesian coordinates, it is required to place the two origins together and make the polar axis coincide with the positive x-axis. So, every point P in the plane will have both cartesian coordinates (x, y) and polar coordinates (r, θ).



It can be seen from the diagram, that

$$x = r\cos\theta \qquad y = r\sin\theta \quad \text{— (1)}$$

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \quad \text{— (2)}$$

To find x and y, we use equation (1) when r & θ are given. To find r and θ, we use equation (2) provided x ~~any~~ and y are given.

Note (1) The signs of x and y determine the quadrant for θ. The angle θ is chosen so that $0 \le \theta < 2\pi$, unless stated otherwise.

(2) When converting from cartesian coordinates to polar, remember that there are infinitely many possible pairs

of polar coordinates representing the same point.

Q Convert ~~the~~ polar coordinate $(\sqrt{2}, \pi/4)$ to cartesian coordinate

Solution: $r = \sqrt{2}$ $\theta = \pi/4$.

$x = r\cos\theta = \sqrt{2}\cos\pi/4 = \sqrt{2}\cdot\frac{1}{\sqrt{2}} = 1$.

$y = r\sin\theta = \sqrt{2}\sin\pi/4 = \sqrt{2}\cdot\frac{1}{\sqrt{2}} = 1$

Therefore, the cartesian coordinates of $(\sqrt{2}, \pi/4)$ are $(1,1)$.

Q Convert the cartesian coordinates $(-1,1)$ to polar coordinates.

Solution: We will first find the polar coordinates (r,θ) of P that satisfy the conditions $r>0$ and $0 \le \theta < 2\pi$. The radial coordinate r of P satisfies the equation

$r^2 = x^2 + y^2 = (-1)^2 + (1)^2 = 2 \Rightarrow r = \sqrt{2}$

The angular coordinate θ of P satisfies the equation

$\tan\theta = \frac{y}{x} = \frac{1}{-1} = -1.$

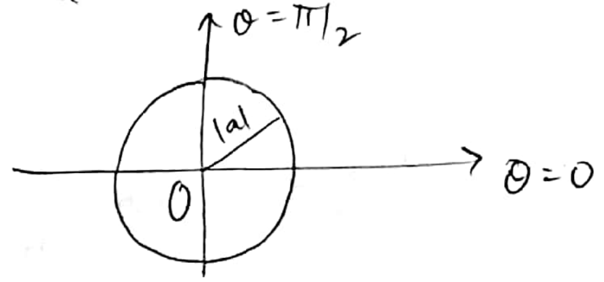
From this and the fact that the point $(-1,1)$ lies in the second quadrant, it follows that the angle satisfying the requirement $0 \le \theta < 2\pi$ is $\theta = \frac{3\pi}{4}$. Thus $(r,\theta) = \left(\sqrt{2}, \frac{3\pi}{4}\right)$ are polar coordinates of P.

Graphing in polar coordinates

Unlike cartesian coordinate system, a point in polar coordinate system has infinitely many different pairs of polar coordinates. However, some of these polar coordinates may satisfy an equation whereas others might not. The graph of an equation $r = f(\theta)$ in polar coordinates is defined to be the set of all points with atleast one pair of coordinates (r, θ) satisfying the ~~coordeon~~ equation.

Q Graph the equation $r = 1$ in polar coordinates.

Solution: The polar equation $r = 1$ is satisfied by the set of all points $(1, \theta)$ where θ is arbitrary. Since the point $(1, \theta)$ is one unit away from the pole, the graph of $r = 1$ is the circle of radius 1 centered at pole. In general, the equation $r = a$ represents a circle of radius $|a|$ centered at pole.

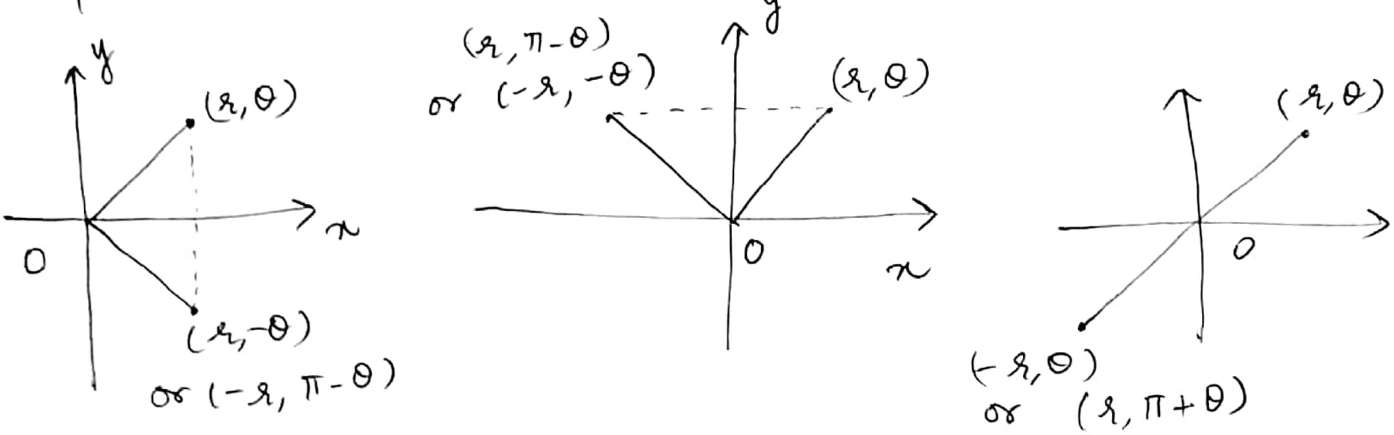


Symmetry Tests for Polar Graphs

(a) Symmetry about the x-axis: A graph in polar coordinates is symmetric about the x-axis if replacing (r, θ) by $(r, -\theta)$, or replacing (r, θ) by $(-r, \pi - \theta)$ in its equation gives an equivalent equation.

(b) Symmetry about the y-axis: A graph in polar coordinates is symmetric about the y-axis if replacing (r, θ) by $(r, \pi-\theta)$, or replacing (r, θ) by $(-r, -\theta)$ in its equation gives an equivalent equation.

(c) Symmetry about the origin: A graph in polar coordinates is symmetric about the origin if replacing (r, θ) by $(-r, \theta)$, or replacing (r, θ) by $(r, \pi+\theta)$ in its equation gives an equivalent equation.

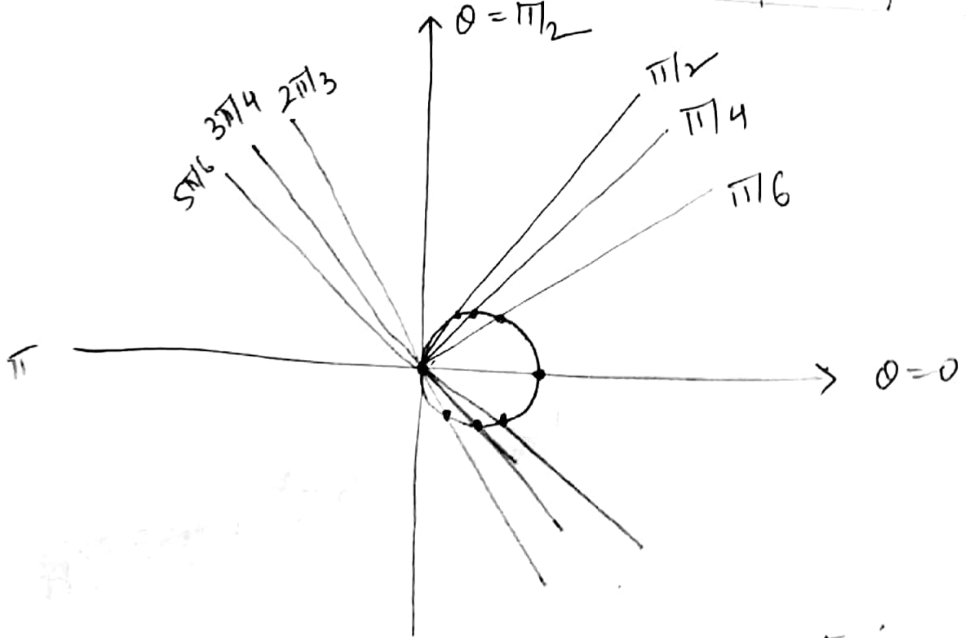


For example, the graph of $r = 1 + \sin\theta$ is symmetric about the y-axis because if (r, θ) satisfies the equation, then $(r, \pi-\theta)$ also satisfies the equation. Similarly, the graph of $r = 3\sin 2\theta$ is symmetric about the x-axis, y-axis & origin.

Q Sketch the graph of the equation $r = 2\cos\theta$ in polar coordinates.

Solution: Notice that replacing θ by $-\theta$ does not change the equation, so the graph is symmetric about polar axis (or x-axis). Therefore, we only need to graph the equation over the interval $[0, \pi]$. The complete graph is obtained by reflecting the points already graphed ~~ove~~ about polar axis.

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
r	2	1.73	1.41	1	0	-1	-1.41	-1.73	-2



On joining these points, the resulting curve is a circle. If we write the given equation in terms of x and y, we get

$$r^2 = 2r\cos\theta \Rightarrow x^2 + y^2 = 2x \Rightarrow x^2 + y^2 - 2x = 0$$

$$\Rightarrow (x-1)^2 + y^2 = 1$$

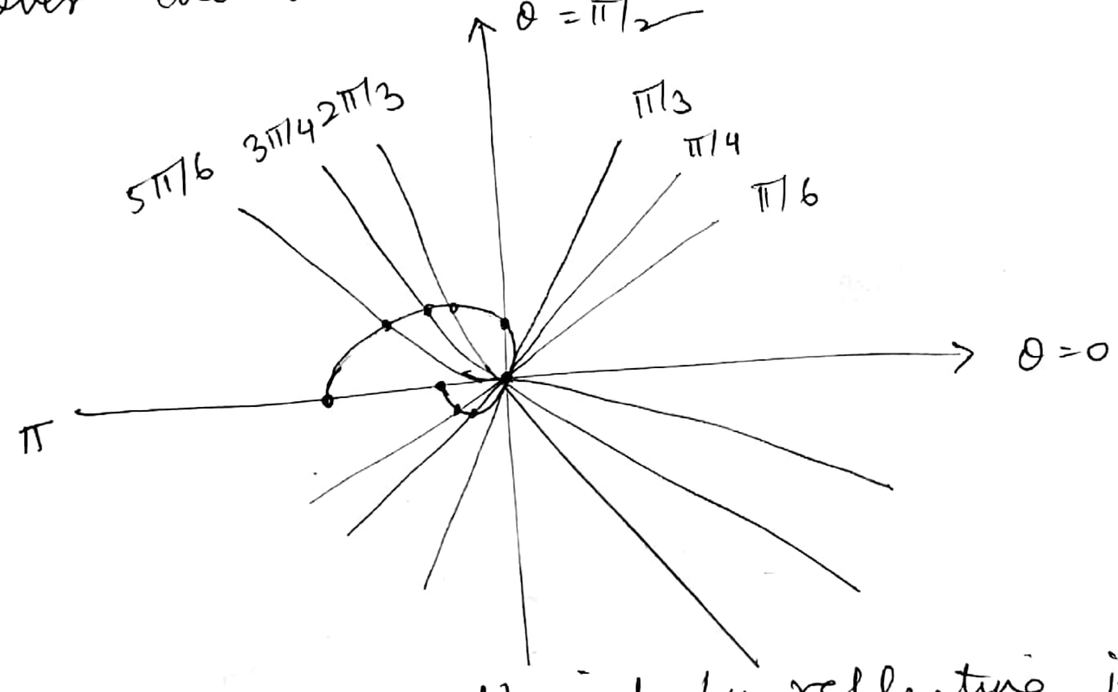
which is a circle of radius 1 centered at (1, 0) in xy-plane.

Q Sketch the graph of $r = 1 - 2\cos\theta$ in polar coordinates.

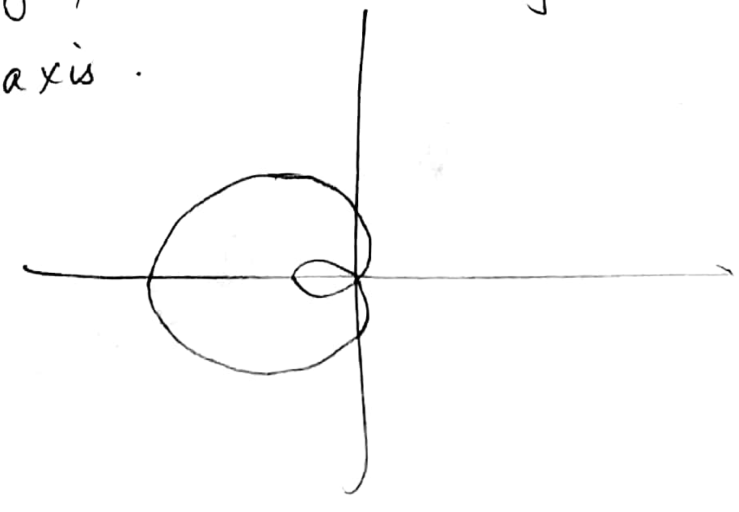
Solution: Replacing θ by $-\theta$ does not alter the equation. so the given graph is symmetric about the polar axis. Therefore, we only need to draw the graph over $[0, \pi]$, other half is symmetrical.

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$r = 1 - 2\cos\theta$	-1	-0.73	-0.41	0	1	2	2.41	2.73	3

when we join these points, we get a portion of the graph over the interval $[0, \pi]$.



The complete graph is obtained by reflecting it about the x-axis.



Q Sketch the curve $r^2 = \cos\theta$ in polar coordinates.

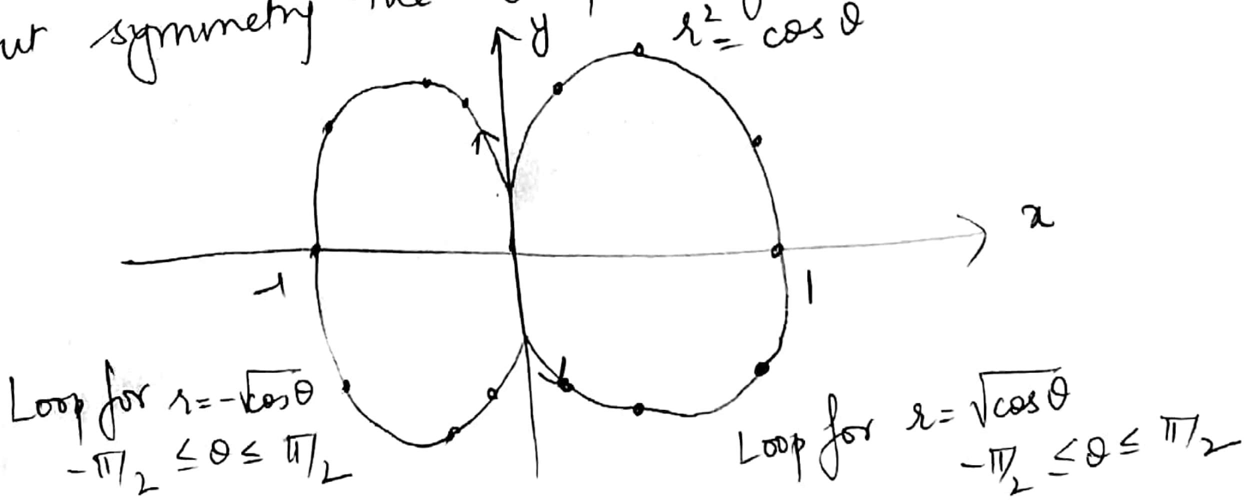
Solution: The curve is symmetric about the x-axis because replacing θ by $-\theta$ does not alter the equation. The curve is also symmetric about the origin because replacing r by $-r$ does not alter the equation. Combining these two symmetries, we get that the curve is symmetric about y-axis also.

The equation $r^2 = \cos\theta$ requires $\cos\theta \geq 0$, so we get the complete graph by varying θ from $-\pi/2$ to $\pi/2$. For each value of θ in the interval between $-\pi/2$ and $\pi/2$, the equation $r^2 = \cos\theta$ gives two values of r:

$$r = \pm\sqrt{\cos\theta}$$

θ	$-\pi/2$	$-\pi/3$	$-\pi/4$	$-\pi/6$	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\cos\theta$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
$r = \pm\sqrt{\cos\theta}$	0	± 0.7	± 0.8	± 0.9	± 1	± 0.9	± 0.8	± 0.7	0

By joining these points & using the information about symmetry the complete graph is obtained.



Q Sketch the graph of $r = \cos 2\theta$ in polar coordinates.

Solution: We observe that replacing θ by $-\theta$ does not change the equation, so the graph is symmetric about polar axis (or x-axis). We first sketch the graph of $r = \cos 2\theta$ by plotting r as a function of θ in the cartesian coordinates.

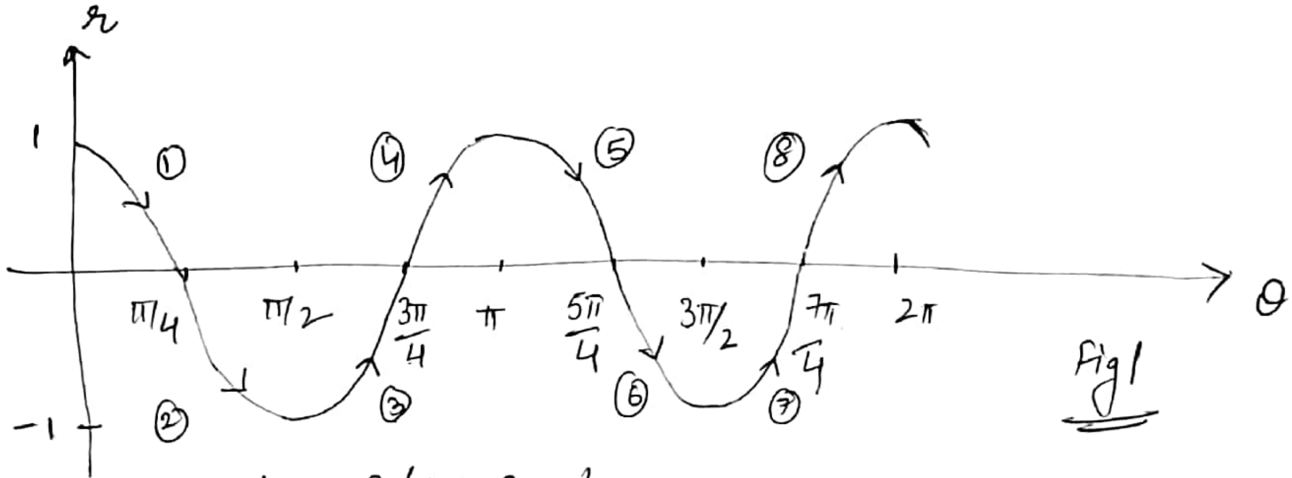


Fig 1

We now draw the polar graph.

→ As θ varies from 0 to $\pi/4$, r varies from 1 to 0. The resulting portion of the polar graph drawn in Fig 2 (indicated by ①).

→ As θ varies from $\pi/4$ to $\pi/2$, r is negative and varies from 0 to -1. The resulting portion of the polar graph is drawn in Fig 2 (indicated by ②). Notice that this portion of the polar curve lies on the opposite side of the pole in the third quadrant, because r is negative.

→ As θ varies from $\pi/2$ to $3\pi/4$, r is still negative and varies from -1 to 0. The resulting portion of the polar graph is drawn in Fig 2 (indicated by ③)

→ As θ varies from $3\pi/4$ to π, r varies from 0 to 1. The resulting portion of the polar graph is drawn in Fig 2 (indicated by 4).

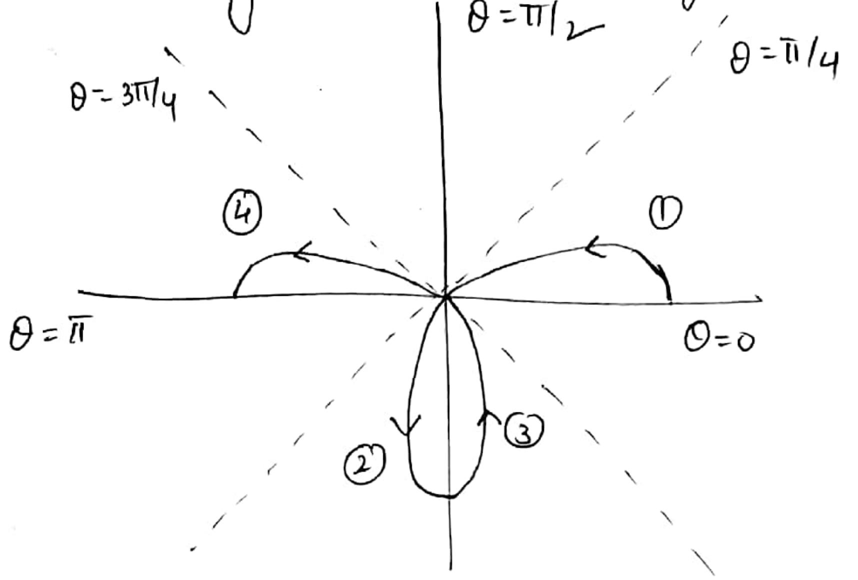


Fig 2

The rest of the curve can be obtained by continuing the preceding analysis from π to 2π, by reflecting the portion already graphed about x-axis.