Unit II . PDE. 22 marks.

Linear PDE of 1st order; Lagrange's Method

Order: Order of a PDE is the max order of the partial derivatives occuring in it.

Degree: degree of highest order partial deri.

$z = f(x,y)$ 1st order partial deri.

$p = \frac{\partial z}{\partial x}$ $q = \frac{\partial z}{\partial y}$

Linear Eqⁿ: A first order eqⁿ $f(x,y,z,p,q)=0$ is stb linear eqⁿ if it is linear in p, q & z
ie of the form $Pp + Qq = Rz + S$ where P,Q,R,S funⁿ of x & y only

Semilinear: $Pp + Qq = R$. P,Q - fun of x & y only & R - fun of x, y & z ($L \Rightarrow SL$) Take $Rz + S = R$

Quasilinear (Lag' form): $Pp + Qq = R$ where P, Q, R - fun of x, y & z

Lagrange's Method

Method for finding a gen solⁿ of quasi linear eqⁿ.
solⁿ of type $\phi(u_1, \dots u_n) = 0$.

Method:
(1) Given $Pp + Qq = R$ - (1)
Consider simultaneous eqⁿ
$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ - (2)
$u_1(x,y,z) = c_1$, $u_2(x,y,z) = c_2$
be 2 indep solⁿ of (2)
Gen solⁿ of (1) is $\phi(u_1, u_2) = 0$

Q $z(xp - yq) = y^2 - x^2$

$P = xz$ $Q = -yz$ $R = y^2 - x^2$

Sim eqⁿ. $\frac{dx}{xz} = \frac{dy}{-yz} = \frac{dz}{y^2 - x^2}$

$1 = 2 \Rightarrow \frac{dx}{x} = -\frac{dy}{y} \Rightarrow xy = c_1$

Taking multipliers x, y, z

$x\,dx + y\,dy + z\,dz = 0 \Rightarrow x^2 + y^2 + z^2 = c_2$

Gen solⁿ is $\phi(xy, x^2 + y^2 + z^2) = 0$.

Q $(y + zx)p - (x + yz)q = x^2 - y^2$

$\frac{dx}{y + zx} = \frac{dy}{-(x + yz)} = \frac{dz}{x^2 - y^2}$

(1) $x, y, -z$
$x^2 + y^2 - z^2 = c_1$

(2) $y, x, 1$
$xy + z = c_2$

$\therefore$ gen solⁿ is $\phi(x^2 + y^2 - z^2, xy + z) = 0$.

$= \quad px(x+y) = qy(x+y) - (x-y)(2x+2y+z)$

$$\frac{dx}{x(x+y)} = \frac{dy}{-y(x+y)} = \frac{dz}{(y-x)(2x+2y+z)}$$

$1=2 \Rightarrow xy = c_1$

Use multipliers $1,1,0$ & $1,1,1$

$$\frac{dx+dy}{x(x+y)-y(x+y)} = \frac{dx+dy+dz}{x(x+y)-y(x+y)+(y-x)(2x+2y+z)}$$

$$\frac{dx+dy}{\cancel{(x-y)}(x+y)} = \frac{dx+dy+dz}{\cancel{(x-y)}(-x-y-z)}$$

$\log(x+y)+\log(x+y+z) = \log c_2 \Rightarrow (x+y)(x+y+z) = c_2$

gen solⁿ $\phi[xy, (x+y)(x+y+z)] = 0$

Q $(mz-ny)p + (nx-lz)q = ly - mx$

$\phi(lx+my+nz, x^2+y^2+z^2) = 0$

① l, m, n ② x, y, z

Q $y^2p - xyq = x(z-2y)$ $\quad \phi(x^2+y^2, yz-y^2) = 0$.

Q $x(y^2-z^2)p + y(z^2-x^2)q - z(x^2-y^2) = 0$

$\phi(x^2+y^2+z^2, xyz) = 0$.

Q $\frac{(b-c)}{a}yzp + \frac{(c-a)}{b}zxq = \frac{(a-b)}{c}xy$

① x, y, z $\quad \phi(ax^2+by^2+cz^2, a^2x^2+b^2y^2+c^2z^2) = 0$.

② ax, by, cz

Q $x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = (x+y)z$

$\phi\left(\frac{x-y}{xy}, \frac{xy}{z}\right) = 0$

$1=2$ $\quad \frac{1}{x}, \frac{1}{y}, \frac{-1}{z}$

Lagrange's form for $\boxed{\text{3 indep vars}}$

let $z = f(x_1, x_2, x_3)$ ie $z \to$ dep on indep vars x_1, x_2, x_3

$p_1 = \frac{\partial z}{\partial x_1}, \quad p_2 = \frac{\partial z}{\partial x_2}, \quad p_3 = \frac{\partial z}{\partial x_3}$

Lagrange's form is $P_1p_1 + P_2p_2 + P_3p_3 = R$ where P_1, P_2, P_3 & R are fnⁿ of x_1, x_2, x_3 & z

Lagrange's eqⁿ : $\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \frac{dx_3}{P_3} = \frac{dz}{R}$

3 indep var ∴ $\boxed{\text{3 indep sol}^n}$

Let $u_1 = c_1$, $u_2 = c_2$, $u_3 = c_3$ be 3 solⁿs

Then gen solⁿ is $\phi(u_1, u_2, u_3) = 0$.

Q Solve $x_1 p_1 + x_2 p_2 + x_3 p_3 = az + \frac{x_1 x_2}{x_3}$.

Lag subeqⁿ are $\frac{dx_1}{x_1} = \frac{dx_2}{x_2} = \frac{dx_3}{x_3} = \frac{dz}{az + \frac{x_1 x_2}{x_3}}$.

(1)=(2) $\log x_1 = \log x_2 + \log c_1 \Rightarrow \frac{x_1}{x_2} = c_1$ —(1)

(2)=(3) $\log x_2 = \log x_3 + \log c_2 \Rightarrow \frac{x_2}{x_3} = c_2$ —(2)

Do not get 3rd solⁿ by equating (1) & (3).

(1)=(4) $\frac{dx_1}{x_1} = \frac{dz}{az + c_2 x_1} \Rightarrow \frac{dz}{dx_1} = \frac{az + c_2 x_1}{x_1} \Rightarrow \frac{dz}{dx_1} - \frac{a}{x_1} \cdot z = c_2$ (linear in z & x_1)

$IF = e^{\int -\frac{a}{x_1} dx_1} = e^{-a \log x_1} = x_1^{-a}$.

$\therefore z \cdot x_1^{-a} = \int c_2 \, x_1^{-a} \, dx_1 = c_2 \cdot \frac{x_1^{1-a}}{1-a} + c_3$.

$\Rightarrow z \cdot x_1^{-a} - \frac{x_2}{x_3} \cdot \frac{x_1^{1-a}}{1-a} = c_3$ —(3)   gen solⁿ $\phi((1),(2),(3)) = 0$.

Charpit's Method

Given a first order PDE $f(x,y,z,p,q) = 0$ —(1)

complete Integral (solⁿ): of a PDE is a relⁿship involving dep & indep vars (but not the partial deri) which satisfies the given PDE & which has as many arb. consts as the no. of indep vars.

→ find partial deri f_x, f_y, f_z, f_p, f_q

→ Substitute in Charpit's Aux. Eqⁿs

$$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-p f_p - q f_q}$$

→ Use any suitable combⁿ (involving p or q or both) to obtain $g(x,y,z,p,q) = 0$ —(2)

→ Solve (1) & (2) to obtain p & q in terms of x, y & z

→ Substitute in $dz = p\,dx + q\,dy$ & integrate

Preferences to obtain (2) : 1st dp & dx or dq & dy
2nd dp & dy or dq & dx
3rd dp & dq.

Find the complete integrals of the eqⁿs :-

Q $(p^2 + q^2) y = qz$ —(1)

$f = p^2 y + q^2 y - qz$

$f_x = 0$, $f_y = p^2 + q^2$, $f_z = -q$, $f_p = 2py$, $f_q = 2qy - z$.

Charpit's Aux Eq^n are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q}.$$

i.e. $$\frac{dp}{-pq} = \frac{dq}{p^2} = \frac{dx}{-2py} = \frac{dy}{-2qy+z} = \frac{dz}{-2p^2y - 2q^2y + qz}$$

dp & dx ✗ | dp & dy ✗

dq & dy ✗ | dq & dx ✗

∴ use (1) & (2).

$$\frac{dp}{-pq} = \frac{dq}{p^2} \Rightarrow pdp + qdq = 0 \Rightarrow p^2 + q^2 = \text{const} = c^2 \quad (2)$$

Using (1) & (2), $c^2 y = qz \Rightarrow q = \frac{c^2 y}{z}$.

Also (2) $\Rightarrow p = \sqrt{c^2 - q^2} = \frac{c}{z}\sqrt{z^2 - c^2y^2}$

$$\therefore dz = pdx + qdy = \frac{c}{z}\sqrt{z^2 - c^2y^2}\,dx + \frac{c^2 y}{z}\,dy$$

$$\Rightarrow \frac{z\,dz - c^2 y\,dy}{\sqrt{z^2 - c^2y^2}} = c\,dx$$

Put $z^2 - c^2y^2 = t$ & integrate

$$\Rightarrow \frac{1}{2}\frac{(z^2 - c^2y^2)^{1/2}}{1/2} = cx + d \Rightarrow (z^2 - c^2y^2) = (cx + d)^2$$

where c & d are arb consts.

Q. $p = (z + qy)^2$ — (1)

$f = (z+qy)^2 - p$

$f_x = 0 \quad f_y = 2(z+qy)q, \quad f_z = 2(z+qy) \quad f_p = -1 \quad f_q = 2(z+qy)y.$

Charpit's Aux eq^n $$\frac{dp}{2p(z+qy)} = \frac{dq}{4q(z+qy)} = \frac{dx}{1} = \frac{dy}{-2y(z+qy)} = \frac{dz}{p - 2qy(z+qy)}$$

(2) = (4) $\Rightarrow \frac{dq}{2q} = \frac{dy}{-y}$ ie $q = \frac{c}{y^2}$ — (2)

Use (2) in (1) $\Rightarrow p = \left(z + \frac{c}{y}\right)^2$

$$dz = pdx + qdy = \left(z + \frac{c}{y}\right)^2 dx + \frac{c}{y^2}dy$$

$$\Rightarrow \frac{dz - \frac{c}{y^2}dy}{\left(z + \frac{c}{y}\right)^2} = dx \Rightarrow \frac{\left(z + \frac{c}{y}\right)^{-1}}{-1} = x + d \Rightarrow z + \frac{c}{y} = -\frac{1}{x+d}$$

where c & d arb const

Q $z = pq$ (dx & dq) Sol^n $z = (x + c_1)(y + c_2)$

Q $2xz - px^2 - 2qxy + pq = 0$ (dq & dy) $[z - ay = b(x^2 - a)]$

Q $2(z + xp + yq) = yp^2$ (dp & dy) $\left[yz = -\frac{a^2}{4x^2} + a\frac{x}{y} + b\right]$

Q. $z^2 = pqxy$ — (1) $f_p = - \quad f_q = - \quad \& \ldots$ (3)

$f = pqxy - z^2$,

$$\frac{dx}{qxy} = \frac{dy}{pxy} = \frac{dz}{2pqxy} = \frac{dp}{2pz - pqy} = \frac{dq}{2qz - pqx}$$

$$\boxed{\frac{p\,dx + x\,dp}{2pxz} = \frac{q\,dy + y\,dq}{2qyz}} \Rightarrow \frac{d(xp)}{xp} = \frac{d(yq)}{yq} \Rightarrow \log xp = \log yq + \log c$$

$\Rightarrow xp = c(yq)$ — (2)

using (1) & (2) $\cancel{p^2 = c}$ $z^2 = (xp)(yq) = c(yq)(yq)$

$\Rightarrow q^2 = \frac{z^2}{cy^2} \Rightarrow q = \frac{z}{ay}$ where $a = \sqrt{c}$

$\therefore p = \frac{az}{x}$ Use $dz = p\,dx + q\,dy$

$\Rightarrow \log z = a \log x + a^{-1} \log y + \log d \Rightarrow z = x^a y^{1/a} . d$ ↳ 2nd constt.

Singular solutions

The singular soln of PDE $f(x,y,z,p,q)=0$ is soln of PDE obtained by eliminating the arb. consts. ~~$c_1, c_2, \ldots c_n$~~ $a \& b$ from the complete soln $\phi(x,y,z,a,b)$ & relation $\frac{\partial \phi}{\partial a} = 0$ & $\frac{\partial \phi}{\partial b} = 0$.

If these 2 are inconsistent, then there is no singular soln.

Q Obtain the ~~complete~~ singular soln of eqn

$2xz - px^2 - 2qxy + pq = 0$ (1)

The complete soln is $z - ay = b(x^2 - a)$.

$\phi = z - ay - bx^2 + ab$

$\frac{\partial \phi}{\partial a} = -y + b = 0$
$\frac{\partial \phi}{\partial b} = -x^2 + a = 0$ $\Rightarrow$ $b = y$ & $a = x^2$

$\therefore \phi = 0 \Rightarrow z - x^2 y - yx^2 + x^2 y = 0$ ie $z = x^2 y$

(Check: $p = 2xy$ $q = x^2$ Put in (1) it will satisfy)

Q $2(z + xp + yq) = yp^2$

comp soln $\phi = yz + \frac{a^2}{4y^2} - \frac{ax}{y} - b = 0$ $\frac{\partial \phi}{\partial a} = \frac{2a}{4y^2} - \frac{x}{y}$ $\frac{\partial \phi}{\partial b} = -1 = 0$ } absurd

$\therefore$ there is no singular soln

Compatible Equations

Two diff eqn are sd to be compatible iff their simultaneous soln for (p & q) makes the eq $dz = p\,dx + q\,dy$ integrable.

Condn for compatibility

Two first order PDE's $f(x,y,z,p,q) = 0$ & $g(x,y,z,p,q) = 0$ are compatible iff

$$\frac{\partial(f,g)}{\partial(x,p)} + \frac{\partial(f,g)}{\partial(y,q)} + p\frac{\partial(f,g)}{\partial(z,p)} + q\frac{\partial(f,g)}{\partial(z,q)} = 0.$$

where $\frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} f_x & f_p \\ g_x & g_p \end{vmatrix}$

2005 Q show that the eqns

$xp - yq = x$ & $x^2 p + q = xz$ are compatible & find their soln

$f = xp - yq - x$ & $g = x^2p + q - xz$

$f_x = p-1$, $f_y = -q$, $f_z = 0$, $f_p = x$, $f_q = -y$

$g_x = 2xp - z$, $g_y = 0$, $g_z = -x$, $g_p = x^2$, $g_q = 1$.

Show compatible.

Sol^n $xp - yq = x$

$(x^2p + q = xz)y$

$(x + yx^2)p = x + xyz$

$\Rightarrow p = \frac{1+yz}{1+xy}$, $q = \frac{x(z-x)}{1+xy}$

$dz = p\,dx + q\,dy = \frac{1+yz}{1+xy}dx + \frac{x(z-x)}{1+xy}dy$

$\Rightarrow (1+xy)dz = dx + yz\,dx + xz\,dy - x^2 dy$

$\Rightarrow (1+xy)dz - z(y\,dx + x\,dy) = dx - x^2 dy$

$\Rightarrow \frac{(1+xy)dz - z(y\,dx + x\,dy)}{(1+xy)^2} = \frac{dx - x^2dy}{(1+xy)^2}$

dividing

$= \frac{-\left(dy - \frac{dx}{x^2}\right)}{\left(\frac{1}{x}+y\right)^2}$ Divide by x^2

$\Rightarrow d\left(\frac{z}{1+xy}\right) = d\left(\frac{1}{\frac{1}{x}+y}\right)$

$\Rightarrow \frac{z}{1+xy} = \frac{1}{\frac{1}{x}+y} + c$ $\Rightarrow z - x = c(1+xy)$

Q Show that the eq^{n}s $xp = yq$ & $z(xp + yq) = 2xy$ are compatible & solve them.

$\boxed{z^2 = xy + c}$

Q Show that the eq^n $z = px + qy$ is compatible with any eq^n $f(x,y,z,p,q) = 0$ ① which is hom. in x, y & z.

Let $g(x,y,z,p,q) = px + qy - z = 0$ ②

As f is hom in x, y & z. $\therefore$ By Euler's Th^m, $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf$

(where n is degree of homogeneity of f)

$g_x = p$, $g_y = q$, $g_z = -1$, $g_p = x$, $g_q = y$

$J_1 = \frac{\partial(f,g)}{\partial(x,p)} = xf_x - pf_p$ $J_2 = \frac{\partial(f,g)}{\partial(y,q)} = yf_y - qf_q$

$J_3 = \frac{\partial(f,g)}{\partial(z,p)} = xf_z + f_p$ $J_4 = \frac{\partial(f,g)}{\partial(z,q)} = yf_z + f_q$

$[f,g] = J_1 + J_2 + pJ_3 + qJ_4 = xf_x + yf_y + f_z(px + qy) = xf_x + yf_y + zf_z$

$= nf = 0$.

$\therefore$ eq^n are compatible.

Scanned with CamScanner

(4)

Q. Show that the eqⁿ $f(x,y,p,q)=0$ & $g(x,y,p,q)=0$ are compatible iff $\frac{\partial(f,g)}{\partial(x,p)}+\frac{\partial(f,g)}{\partial(y,q)}=0$. —(1)

z-missing ∴ $f_z = g_z = 0$.

compatibility condⁿ ~~reduces to~~ $\frac{\partial(f,g)}{\partial(x,p)}+\frac{}{\partial(y,q)}+p\frac{}{\partial(z,p)}+q\frac{}{\partial(z,q)}=0$

$\frac{\partial(f,g)}{\partial(z,p)}=f_z g_p - f_p g_z = 0$, $|||^{ly}$ $\frac{\partial(f,g)}{\partial(z,q)}=0$. ∴ reduces to (1).

Q verify that eqⁿ $p=P(x,y)$ & $q=Q(x,y)$ are compatible iff $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$.

let $f=P(x,y)-p$, $g=Q(x,y)-q$.

$f_x=P_x$, $f_y=P_y$, $f_z=0$, $f_p=-1$, $f_q=0$

$g_x=Q_x$, $g_y=Q_y$, $g_z=0$, $g_p=0$, $g_q=-1$.

compⁿ condⁿ is ? ______ $=0$. —(1)

$J_1=Q_x$ $J_2=-P_y$ $J_3=0$ $J_y=0$.

∴ (1) becomes. $Q_x-P_y=0$ ie $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0$.

Practice Questions ~~eqs~~ (Partial Diff Eqⁿ) ~~marks~~

Q1 Find the general solⁿ of (10)

$x(x^2+3y^2)p-y(3x^2+y^2)q=2z(y^2-x^2)$.

Mult: $\frac{1}{x},\frac{1}{y},\frac{-1}{z}$ & $x,y,0$ solⁿ $\phi\left(\frac{xy}{z},(x^2+y^2)z\right)=0$.

Q2 Find the complete integral of (10)

$(p+y)^2+(q+x)^2=1$ dx & dq $z+xy=(\sqrt{1-c_1^2})x+c_1y+c_2$

Q3 Reduce the eqⁿ (15)

$y^2\frac{\partial^2 z}{\partial x^2}-2xy\frac{\partial^2 z}{\partial x\partial y}+x^2\frac{\partial^2 z}{\partial y^2}=\frac{y^2}{x}\frac{\partial z}{\partial x}+\frac{x^2}{y}\frac{\partial z}{\partial y}$ to canonical form & hence solve it.

Parabolic $\lambda=\frac{x}{y}$ $u=x^2+y^2$ $v=x^2-y^2$ $z=\phi_1(u)v+\phi_2(u)$.

Classification of 2nd order PDE

Review $z = f(x,y)$

1st order PDE $p = \frac{\partial z}{\partial x}$ $\quad q = \frac{\partial z}{\partial y}$

2nd order PDE $\quad p, q \to$ same. $\quad r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}, \quad t = \frac{\partial^2 z}{\partial y^2}$.

$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$.

$z < {x \atop y}$.

$\rightarrow z = x^2 + f(y) \Rightarrow \frac{\partial z}{\partial x} = 2x$

$\int \frac{\partial z}{\partial x} dx = \int 2x \Rightarrow z = x^2 + f(y) \to$ fun of y (not const)

$\rightarrow \frac{\partial^2 z}{\partial x^2} = 0$ Integrate $\int \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \int 0 \Rightarrow \frac{\partial z}{\partial x} = f(y) \xrightarrow{\text{Integrate}} z = f(y).x + g(y)$.

$\rightarrow z < {u < {x \atop y} \atop v < {x \atop y}}$.

Then $\boxed{\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial x} \\ \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial y} \end{aligned}}$

Also, $\frac{\partial z}{\partial u} < {u < {x \atop y} \atop v < {x \atop y}}$

Illy $\frac{\partial z}{\partial v}$.

$$\boxed{\begin{aligned} \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) &= \frac{\partial^2 z}{\partial u^2}.\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial v \partial u}.\frac{\partial v}{\partial x} \\ \text{Illy } \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) &= \frac{\partial^2 z}{\partial u^2}.\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial v \partial u}.\frac{\partial v}{\partial y} \\ \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right) &= \frac{\partial^2 z}{\partial u \partial v}.\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial v^2}.\frac{\partial v}{\partial x} \\ \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right) &= \frac{\partial^2 z}{\partial u \partial v}.\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial v^2}.\frac{\partial v}{\partial y} \end{aligned}}$$

Canonical forms

Consider the 2nd order PDE

$R\frac{\partial^2 z}{\partial x^2} + S\frac{\partial^2 z}{\partial x \partial y} + T\frac{\partial^2 z}{\partial y^2} = V$

where R, S, T are fun of x, y

& V is fun of x, y, z, p, q.

It is called hyperbolic iff $S^2 - 4RT > 0$

Parabolic $\quad S^2 - 4RT = 0$

Elliptic $\quad S^2 - 4RT < 0$.

Steps! [Given PDE & $u = f(x,y)$, $v = g(x,y)$

$x = \phi(u,v)$, $y = \psi(u,v)$

→ use transf deri $\frac{\partial u}{\partial x}$ & ---

to transform PDE into $z < {u \atop v}$

ie involving terms $\frac{\partial z}{\partial u}, \ldots$]

Q Transform the eqn $\frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial^2 z}{\partial y^2}$ — (1) by transf $u = y + \frac{x^2}{2}$, $v = y - \frac{x^2}{2}$

(ie. transf. into another PDE with z-dep & u, v — indep var)

$u = y + \frac{x^2}{2}$

$v = y - \frac{x^2}{2}$

$\frac{u+v}{2} = y$

$\therefore y = \frac{u+v}{2}$

& $x^2 = u - v$.

$\frac{\partial u}{\partial x} = x \quad \frac{\partial u}{\partial y} = 1, \quad \frac{\partial v}{\partial x} = -x, \quad \frac{\partial v}{\partial y} = 1.$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}(x) + \frac{\partial z}{\partial v}(-x) = x\frac{\partial z}{\partial u} - x\frac{\partial z}{\partial v}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left[x\frac{\partial z}{\partial u} - x\frac{\partial z}{\partial v}\right]$$

$$= x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial z}{\partial u} - x\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial v}\right] - \frac{\partial z}{\partial v} \quad \text{(Product rule)}$$

$$= x\left[\frac{\partial^2 z}{\partial u^2}\cdot\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial v.\partial u}\cdot\frac{\partial v}{\partial x}\right] + \frac{\partial z}{\partial u} - x\left[\frac{\partial^2 z}{\partial u\partial v}\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial v^2}\cdot\frac{\partial v}{\partial x}\right] - \frac{\partial z}{\partial v}$$

$$= x\left[\frac{\partial^2 z}{\partial u^2}.x + \frac{\partial^2 z}{\partial v.\partial u}.(-x)\right] + \frac{\partial z}{\partial u} - x\left[\frac{\partial^2 z}{\partial u\partial v}.x + \frac{\partial^2 z}{\partial v^2}(-x)\right] - \frac{\partial z}{\partial v}$$

$$= \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + \cancel{x^2\frac{\partial^2 z}{\partial u^2}} + \cancel{x^2\frac{\partial^2 z}{\partial v^2}} - 2x^2\frac{\partial^2 z}{\partial u\partial v}$$

Illy $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right] = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)$$

$$= \left[\frac{\partial^2 z}{\partial u^2}\cdot\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial v\partial u}\cdot\frac{\partial v}{\partial y}\right] + \left[\frac{\partial^2 z}{\partial u\partial v}\cdot\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial y}\right]$$

$$= \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial v\partial u} + \frac{\partial^2 z}{\partial v^2}$$

$\therefore x^2\frac{\partial^2 z}{\partial y^2} = \cancel{x^2\frac{\partial^2 z}{\partial u^2}} + 2x^2\frac{\partial^2 z}{\partial v\partial u} + \cancel{x^2\frac{\partial^2 z}{\partial v^2}}$

Substituting in (1),

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} - 2x^2\frac{\partial^2 z}{\partial u\partial v} = 2x^2\frac{\partial^2 z}{\partial u\partial v}$$

$$\Rightarrow \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = 4x^2\frac{\partial^2 z}{\partial u\partial v}$$

$$= 4(u-v)\frac{\partial^2 z}{\partial u\partial v} \quad \Rightarrow \frac{\partial^2 z}{\partial u\partial v} = \frac{1}{4(u-v)}\left[\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right].$$

Now introduce hyp, para, ellipse. ← see fig!

(1) Hyperbolic: Given PDE s.t. we have $S^2-4RT>0$

steps: (i) consider the eqn $R\lambda^2 + S\lambda + T = 0$ (in λ)

let $\lambda_1(x,y)$ & $\lambda_2(x,y)$ be its roots ($\because$ R, S, T – fun of x, y)

(ii) consider d.e. $\frac{dy}{dx} + \lambda_1(x,y) = 0$

& $\frac{dy}{dx} + \lambda_2(x,y) = 0$

let their soln be $f(x,y) = $ constt & $g(x,y) = $ constt resp

(iii) let $u = f(x,y)$ & $v = g(x,y)$

& transform the given D.E accordingly

Q Reduce the eq^n $\frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial^2 z}{\partial y^2}$ to canonical form.

$R=1,\ S=0,\ T=-x^2$

$S^2-4RT = 4x^2 > 0$ $\therefore$ Hyperbolic.

step (i) Consider $R\lambda^2+S\lambda+T=0$ i.e. $\lambda^2-x^2=0 \Rightarrow \lambda = \pm x$

let $\lambda_1(x,y) = x$ & $\lambda_2(x,y) = -x$

(ii) $\frac{dy}{dx} + \lambda_1(x,y) = 0 \Rightarrow \frac{dy}{dx} + x = 0 \Rightarrow dy + x\,dx = 0$

$\Rightarrow y + \frac{x^2}{2} = c_1$

$\frac{dy}{dx} + \lambda_2(x,y) = 0 \Rightarrow y - \frac{x^2}{2} = c_2$

(iii) let $u = y + \frac{x^2}{2}$, $v = y - \frac{x^2}{2}$

Now proceed as in previous question

Parabolic equation $S^2-4RT=0$

(i) Consider $R\lambda^2+S\lambda+T=0$ $\lambda_1=\lambda_2=\lambda$ (equal roots)

(ii) Solve $\frac{dy}{dx} + \lambda(x,y) = 0 \Rightarrow g(x,y)=0$

(iii) let $u = g(x,y)$ & v = any suitable function of x & y & transform.

Q Reduce the equation $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$

$R=1,\ S=2,\ T=1$ $S^2-4RT=0$ $\therefore$ parabolic

(i) $R\lambda^2+S\lambda+T=0 \Rightarrow \lambda^2+2\lambda+1=0 \Rightarrow \lambda = -1,-1$ (equal roots)

(ii) $\frac{dy}{dx} + \lambda = 0 \Rightarrow \frac{dy}{dx} - 1 = 0 \Rightarrow y - x =$ constant

let $u = y-x$ & $v = y+x$ & transform

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = -\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left[-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right] = -\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= -\left[\frac{\partial^2 z}{\partial u^2}\cdot\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial v \partial u}\cdot\frac{\partial v}{\partial x}\right] + \left[\frac{\partial^2 z}{\partial u \partial v}\cdot\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial v^2}\cdot\frac{\partial v}{\partial x}\right]$$

$$= \frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

Similarly, $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = -\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right).$$

$$= \left[\frac{\partial^2 z}{\partial u^2} \cdot \frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial v \cdot \partial u} \cdot \frac{\partial v}{\partial y}\right] + \left[\frac{\partial^2 z}{\partial u \partial v} \cdot \frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial v^2} \cdot \frac{\partial v}{\partial y}\right]$$

$$= \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

Substituting these values in the given equation, we obtain

$4\frac{\partial^2 z}{\partial v^2} = 0 \Rightarrow \frac{\partial^2 z}{\partial v^2} = 0$. On integrating, $\frac{\partial z}{\partial v} = \phi_1(u)$

again integrating, $z = \phi_1(u).v + \phi_2(u)$

Therefore, $z = [\phi_1(y-x)](y+x) + \phi_2(y-x)$

Elliptic

$S^2 - 4RT < 0$

Consider $R\lambda^2 + S\lambda + T = 0$. Since $S^2 - 4RT < 0$

therefore we get complex roots and hence complex solutions of the form

$\phi(x,y) + i\psi(x,y) = \text{constant}$.

Let $u = \phi(x,y)$ & $V = \psi(x,y)$ & now transform

Q Reduce the equation $\frac{\partial^2 z}{\partial x^2} + x^2\frac{\partial^2 z}{\partial y^2} = 0$ to canonical form.

$R = 1 \quad S = 0 \quad T = x^2$ $\qquad S^2 - 4RT = -4x^2 < 0$

$\therefore$ Elliptic equation

let $\lambda_1(x,y) = ix$ and $\lambda_2(x,y) = -ix$ which are complex roots of the equation $\lambda^2 + x^2 = 0$.

Now, solving $\frac{dy}{dx} + \lambda_1(x,y) \Rightarrow y + \frac{ix^2}{2} = \text{constant}$

$$\frac{dy}{dx} + \lambda_2(x, y) = 0 \quad \Rightarrow \quad y - \frac{x^2}{2} = \text{constant}$$

let $u = y$ & $v = x^2$ & now transform.

The transformed equation is

$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = \frac{-1}{2v}\frac{\partial z}{\partial v}.$$