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B.Sc (PS)Computer Science VI Sem

Dielectrics

This equation corresponds to a medium having one absorption band. The existence of several absorption bands is explained by postulating the existence of particles with several different natural frequencies. In this case equation (5) takes the form

$$n^2 = 1 + \sum_p \frac{A_p d^2}{d^2 - d_p^2} \quad (6)$$

where A_p is proportional to the number of particles per unit volume whose natural freq. corresponds to d_p . Figure below shows the form of dispersion curve by Sellmeier's formula in case where there are two absorption bands.

Although Sellmeier's formula represents a considerable improvement over Cauchy's formula, it fails in the region of absorption band, because it indicates that n is ∞ and discontinuous at d_1, d_2 etc.

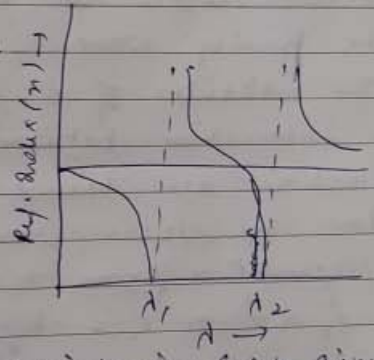


Fig: Dispersion Curve given by Sellmeier's formula for absorption bands.

However in regions remote from an absorption band it gives better results than Cauchy's formula having same no. of constants, and it can be seen that Sellmeier's formula reduces to Cauchy's formula when certain approximations are made.

If there is one absorption band and if $d_0 \ll d$, then

eq. (5) gives

$$n^2 = 1 + A \left[1 - \frac{d_0^2}{d^2} \right]^{-2} = 1 + A + \frac{A d_0^2}{d^2} + \frac{A d_0^4}{d^4} \quad (7)$$

d^2 same as Cauchy's formula

This equation corresponds to a medium having one absorption band. The existence of several absorption bands is explained by postulating the existence of particles with several different natural frequencies. In this case equation (5) takes the form

$$n^2 = 1 + \frac{\epsilon}{P} \frac{A_p d^2}{d^2 - d_p^2} \quad (6)$$

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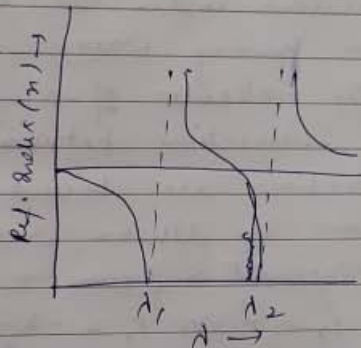


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However in regions remote from an absorption band it gives better results than Cauchy's formula having same no. of constants, and it can be seen that Sellmeier's formula reduces to Cauchy's formula when certain approximations are made.

If there is one absorption band and if $d_0 \ll \lambda$, then

Eq. (5) gives

$$n^2 = 1 + A \left[1 - \frac{d_0^2}{\lambda^2} \right]^{-2} = 1 + A + \frac{A d_0^2}{\lambda^2} + \frac{A d_0^4}{\lambda^4} \quad (7)$$

λ^2 same as Cauchy's formula

Derivation of Lorentz's formula / Dispersion in Gases

The ~~atom~~ ^{atomic} theory of dispersion ~~given~~ based on electromagnetic theory was given by H.A. Lorentz. In case of gases, he assumed:

1. When an electromagnetic wave passes through a medium, the atomic (or molecular) dipoles are created in the medium due to relative displacement of electrons and nuclei of neutral atoms due to electric field of the wave. Thus a dipole moment is induced in the gas molecule.
2. The atomic (or molecular) dipoles execute forced vibrations under the oscillatory electric field of the passing wave.
3. The molecules of a gas are far apart; therefore the interactions between them may be neglected.
4. Electrons are bound to the nucleus in an atom by linear restoring forces proportional to displacement r from the equilibrium position given by $F_r \propto r$ or $F_r = -Kr$, where K is a certain positive constant which depends on the structure and properties of particles.
5. The electrons experience a damping or frictional force proportional to velocity to account for the absorption; given by

$$F_d = -\alpha \frac{dr}{dt}$$
 where α is a +ve constant of proportionality.
6. The electric field E is constant in space over an atom or molecule, thus

$$E = E_0 e^{i(kr - \omega t)} = E_0 e^{-i\omega t}$$

In the light of above assumptions, the total force acting on the electron:

Force $\vec{F} = e\vec{E} - k\vec{r} - \alpha \frac{d\vec{r}}{dt}$

hence according to Newton second law, the equation of motion of the electron is

$$m \frac{d^2 \vec{r}}{dt^2} = e\vec{E} - k\vec{r} - \alpha \frac{d\vec{r}}{dt}$$

or $\frac{d^2 \vec{r}}{dt^2} + \frac{\alpha}{m} \frac{d\vec{r}}{dt} + \frac{k}{m} \vec{r} = \frac{q}{m} \vec{E}$

or $\frac{d^2 \vec{r}}{dt^2} + \gamma \frac{d\vec{r}}{dt} + \omega_0^2 \vec{r} = \frac{q}{m} \vec{E}$ — (1)

where $\gamma = \frac{\alpha}{m}$ and $\omega_0^2 = \frac{k}{m}$

The solution of eq. (1) is given by

$$\vec{r} = \frac{q}{m} \frac{\vec{E}_0 e^{-i\omega t}}{(\omega_0^2 - \omega^2 - i\gamma\omega)} \quad \text{--- (2)}$$

The Polarisation \vec{P} is equal to dipole moment ($q\vec{r}$) per unit volume so that if there are N dispersion electrons per unit volume, we may write

$$\begin{aligned} \vec{P} &= Nq\vec{r} \\ &= \frac{Nq \times \frac{q}{m} E_0 e^{-i\omega t}}{\omega_0^2 - \omega^2 - i\gamma\omega} = \frac{Nq^2 E}{m(\omega_0^2 - \omega^2 - i\gamma\omega)} \quad \text{--- (3)} \end{aligned}$$

In the derivation of eq. (3) it is assumed that all the oscillators have the same natural frequency which shows that the medium absorbs only one spectral line. To account for other spectral lines, it is supposed that there are N molecules per unit volume and in each molecule there are f_1 oscillators whose constants are ω_1 and γ_1 ; f_2

Oscillators when constants are ω_0 and γ_0 and so on. The generalization of eq. (3) can be written as

$$\vec{P} = Nq \sum_k \vec{x}_k$$

$$= \frac{Nq^2 \vec{E}}{m} \sum_k \frac{f_k}{(\omega_k^2 - \omega^2 - i\gamma_k \omega)} \quad (4)$$

The electrical polarizability $\vec{\alpha}$ is given by

$$\vec{\alpha} = \frac{\vec{P}}{\vec{E}} = \frac{Nq^2}{m} \sum_k \frac{f_k}{(\omega_k^2 - \omega^2 - i\gamma_k \omega)} \quad (5)$$

$$\text{The dielectric constant} = 1 + \frac{1}{\epsilon_0} \frac{\vec{P}}{\vec{E}}$$

$$= 1 + \frac{1}{\epsilon_0} \frac{Nq^2}{m} \sum_k \frac{f_k}{(\omega_k^2 - \omega^2 - i\gamma_k \omega)} \quad (6)$$

Sq. (6) shows that the dielectric constant is complex. we thus conclude that refractive index is also complex because refractive index is equal to the square root of dielectric constant.

Hence $n^2 = 1 + \frac{1}{4\pi\epsilon_0} \cdot \frac{4\pi Nq^2}{m} \sum_k \frac{f_k}{(\omega_k^2 - \omega^2 - i\gamma_k \omega)}$ (7)

(Lorentz-dispersion formula)

This is known as dispersion formula. The significance of the complex refractive index is that there is an absorption of energy in the medium. The refractive index has become frequency dependent. Now we consider the following situations:

(a) Normal dispersion: In the region remote from the natural frequencies of oscillators i.e. the absorption frequencies of the medium, the term $\gamma_k \omega$ in the denominator of eq. (7) is very small, so can be neglected.

$$n^2 = 1 + \frac{1}{4\pi\epsilon_0} \cdot \frac{4\pi Nq^2}{m} \sum_k \frac{f_k}{(\omega_k^2 - \omega^2)} \quad (8)$$

The above expression shows that the refractive index is real and increases with frequency of the incident wave. This is normal dispersion.

We know that $\omega = 2\pi f = \frac{2\pi c}{\lambda}$

Substituting the value of ω in equation (8), we have

$$n^2 = 1 + \frac{Nq^2}{m\epsilon_0} \sum_k \frac{f_k}{\omega^2 - \omega_k^2}$$

$$\text{or } n^2 = 1 + \frac{Nq^2}{m\epsilon_0 \cdot (4\pi^2 c^2)} \sum_k \frac{f_k \lambda^2 \lambda_k^2}{(\lambda^2 - \lambda_k^2)}$$

$$\text{or } n^2 = 1 + \sum_k \frac{A_k \lambda^2}{(\lambda^2 - \lambda_k^2)} \quad \text{--- (9)}$$

where $A_k = \frac{Nq^2 A_k f_k}{m\epsilon_0 \times 4\pi^2 c^2}$

Eq. (9) is known as Sellmeier's Equation

→ If we assume that $\lambda \gg \lambda_k$ then eq. (9) can be written as

$$n^2 = 1 + \sum_k A_k \left(1 - \frac{\lambda_k^2}{\lambda^2}\right)^{-1}$$

$$= 1 + \sum_k A_k + \sum_k \frac{A_k \lambda_k^2}{\lambda^2} + \sum_k \frac{A_k \lambda_k^4}{\lambda^4} + \dots$$

$$\text{or } n^2 = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4} + \dots \quad \text{--- (10)}$$

where $A = 1 + \sum_k A_k$; $B = \sum_k A_k \lambda_k^2$; $C = \sum_k A_k \lambda_k^4$

Eq. (10) is called Cauchy's dispersion formula.
 Co-efficient A is the coefficient of refraction and
 Co-efficient B is the coefficient of dispersion.

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Substituting the value of ω in equation (8), we have

$$n^2 = 1 + \frac{Ng^2}{m\epsilon_0} \sum_k \frac{fk}{\frac{4\pi^2 c^2}{\lambda^2} - \frac{4\pi^2 c^2}{\lambda_k^2}}$$

or $n^2 = 1 + \frac{Ng^2}{m\epsilon_0 \cdot (4\pi^2 c^2)} \sum_k \frac{fk \lambda^2 \lambda_k^2}{(\lambda^2 - \lambda_k^2)}$

or $n^2 = 1 + \sum_k \frac{A_k \lambda^2}{(\lambda^2 - \lambda_k^2)}$ ——— (9)

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Optical Phenomena:-

It is well known that some solids are transparent and some are opaque to the light. Further more, some solid surfaces are known to be highly reflecting while others bend the e.m waves incident on them. These properties are investigated by measuring certain physical quantities that have a close relationship with dielectric constant $\epsilon_r(\omega)$ of the solid. This dielectric constant represents the behaviour of all electrons, those in filled and those in unfilled bands. Mathematical relations applied for the study of optical of solids follow from the theory of propagation of e.m waves. An account of the same is summarized below.

The electric field associated with the e.m waves travelling along z-direction in a medium is expressed as

$$E = E_0 \exp \left[-i\omega \left(t - n^*(\omega)z/c \right) \right] \quad \text{--- (1)}$$

where $n^*(\omega)$ is the complex index of refraction described as

$$n^*(\omega) = n + ik = \sqrt{\epsilon_r(\omega)} \quad \text{--- (2)}$$

with the usual definition

$$n^*(\omega) = c/v \quad \text{--- (3)}$$

where v is the velocity in medium.

We know that $\epsilon_r^*(\omega) = \epsilon_r'(\omega) + i\epsilon_r''(\omega)$

Then using eq. (2) we get

$$\epsilon_r'(\omega) = n^2 - k^2$$

$$\text{and } \epsilon_r''(\omega) = 2nk \quad \text{--- (4)}$$

where n & k are two optical constants (and hence the dielectric constant, $\epsilon_r(\omega)$).

Application to Plasma:-

A medium of equal conc of +ve and -ve charges, with at least one charge type being mobile, is known as a Plasma. A metallic crystal may be described as an assembly of immovable +ve ion cores (atomic nuclei) and conduction electrons that are nearly free to move over the whole of the crystal. The condition of charge neutrality is ~~not~~ maintained because of the balance struck between the negative charge on conduction electrons and an equal concentration of +ve charge on ion cores. Thus a metal serves as a good ex. of plas.

(a) Plasma Oscillations, Plasma frequency & Plasmas:

An idealized plasma in which both particles carrying +ve and -ve charges, respectively are motionless is referred to as cold plasma, but every real plasma at temperatures of interest is ~~characterized~~ characterized by thermal agitation which at times are random in nature. A random motion may be a momentary fluctuation in the equilibrium position of an electron, caused by the average electrostatic field of all other electrons.

The position fluctuation would create a charge imbalance in the region of that electron, as a result of which other electrons would rush into that region in order to restore the condition of charge neutrality.

At any finite temperature, electrons being very light particles, move with fairly large speed relative to ions which we consider to be at rest.

The electrons rushing into the region of the electron that suffered a fluctuation in its equilibrium position are unable to stop at the desired

positions and overshoot their mark on account of the large kinetic energy which represents $\frac{1}{2}$ the total energy there. As soon as, energy goes totally electrostatic, electrons turn around and attempt again to approach the wanted locations in the region of the misbehaving e's. The repetition of this process constitutes the collective oscillatory motion. These collective oscillations are called plasma oscillations.

→ Calculate the dielectric constant of a plasma in the absence of an external source of perturbation.

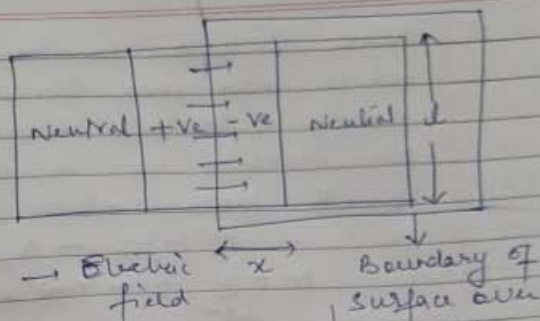
Consider 2-dim. model of metal plasma. On account of a random fluctuation, in the equilibrium positions of electrons, two adjoining regions of +ve and -ve charge densities are created, meaning thereby that the charge neutrality condition is destroyed in these regions. In order to calculate the frequency of a given plasma electron, we work with a picture that is alternative to the one used earlier for describing the plasma oscillations.

Let a small volume element enclosed by a surface S in the negatively charged region have a charge q , the electric field in this small volume is given by Gauss theorem:

$$\int_S \vec{E} \cdot d\vec{S} = q/\epsilon_0 \quad \text{--- (1)}$$

And ~~suppressing~~ suppressing the third spatial axis, we have

$$\int_S \vec{E} \cdot d\vec{S} = -lE \quad (\text{see fig. 1}) \quad \text{--- (2)}$$



Also $q = -lxN_0e$ — (3) which $\int \vec{E} \cdot d\vec{s}$ is taken.
 where x denotes the displacement or the overshoot in the direction of E .

From the above three relations we have

$$\vec{E} = \frac{N_0 e x}{\epsilon_0} \quad (4)$$

Electric field \vec{E} serves as a perturbation and drives an electron into oscillatory motion.

Equation of motion has the form

$$m \frac{d^2 x}{dt^2} = -eE \quad (5)$$

Using (4) in (5) we get

$$\frac{d^2 x}{dt^2} = -\frac{N_0 e^2}{m \epsilon_0} x \quad (6)$$

This relation describes a simple harmonic motion of the characteristic frequency

$$\omega_p = \left(\frac{N_0 e^2}{m \epsilon_0} \right)^{1/2} \quad (7)$$

where ω_p is known as plasma freq.
 In the presence of an external field, both \vec{E} and x are bound to a common oscillatory character represented by the time-dependant perturbation $\propto \exp(-i\omega t)$, ω angular freq of the perturbation force.

Now dielectric constant of a plasma in which +ve ions cores are at rest can be easily calculated. Under the influence of E , an electron has a dipole moment and the bulk polarization of plasma is

$$P(\omega) = -N_0 e x \quad \text{--- (8)}$$

$$m \frac{d^2 x}{dt^2} = -m \omega^2 x = -e E(\omega) \quad \text{--- (9)}$$

$$\begin{aligned} \text{But } \epsilon_r(\omega) &= \frac{D(\omega)}{\epsilon_0 E(\omega)} = \frac{\epsilon_0 E(\omega) + P(\omega)}{\epsilon_0 E(\omega)} \\ &= 1 + \frac{P(\omega)}{\epsilon_0 E(\omega)} \quad \text{--- (10)} \end{aligned}$$

Using (8) & (9), we have

$$\epsilon_r(\omega) = 1 - \frac{N_0 e^2}{m \epsilon_0 \omega^2} \quad \text{--- (11)}$$

$$\text{or } \boxed{\epsilon_r(\omega) = 1 - \frac{\omega_p^2}{\omega^2}} \quad \text{(using eq. 7) --- (12)}$$

exp. of Dielectric constant for free electron gas when $\omega = \omega_p$ then $\epsilon_r(\omega) = 0$, this condition refers to the longitudinal plasma oscillations whose wavevector is taken as nearly zero. The energy quanta $\hbar \omega_p$ are commonly known as plasmons.

During longitudinal plasma oscillations, the e^- gas is moved as whole relative to +ve ions.

Fig 2. shows how the regions of -ve charge and +ve charge emerge out of a neutral charge distribution within a thin metal slab.

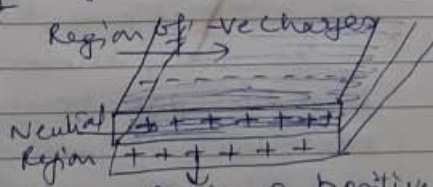


Fig 2: The negative charges (electrons) against the background of +ve ions within metal slab.

→ The Plasmons may be excited by passing e^- thru a thin metal film. The reflection of e^- or photon is also used for the purpose. During the process the charge of an electron coupled with the electrostatic field fluctuations of plasma oscillations and the reflected or transmitted electron shows a loss of energy that equals an integral multiple of $\hbar\omega_p$.

Transverse optical modes in Plasma:-

First dispersion relation for the e-m waves, taken as a model transverse wave will be derived and then, using the above theory, a relation for the positive real values of the dielectric constant will be obtained. Thus derived relation will describe the e-m waves that propagate through the plasma, establishing thereby the excitation of the transverse optical modes of the plasma oscillation.

The e-m wave equation in a non-magnetic isotropic medium is given by

$$\text{Mott} \epsilon_r(\omega, \vec{k}) \frac{\partial^2 \vec{E}}{\partial t^2} = \nabla^2 \vec{E} \quad (13)$$

where \vec{k} is the wave vector of the e-m wave, with the electric field vector \vec{E} varying as $-\exp(i\vec{k} \cdot \vec{r}) \exp(-i\omega t)$, we get the following dispersion relation:

$$\epsilon_r(\omega, \vec{k}) \text{Mott} \omega^2 = k^2 \quad (14)$$

substituting the value of the dielectric constant from eq. (12) in eq. (14) we get-

$$\omega^2 \left(1 - \frac{\omega_p^2}{\omega^2} \right) = \frac{k^2}{\text{Mott}} \quad (15)$$

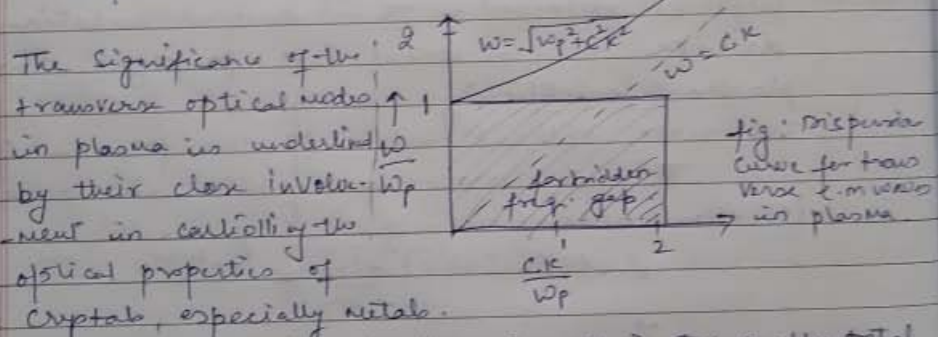
→ for $\omega < \omega_p$ - the above relation gives a negative value of the dielectric constant and an imaginary value of the wave vector. It gives k -dependence of waves & these waves cannot propagate thru the medium and suffer complete reflection.

→ for $\omega > \omega_p$, eq. (15) gives a +ve real value of the dielectric const. and real k -values. Thus the medium is transparent to these waves. we can rewrite eq. (15) as

$$\omega^2 = \omega_p^2 + \frac{c^2 k^2}{\mu_0 \epsilon_0} \quad (16)$$

$$\text{or } \omega^2 = \omega_p^2 + c^2 k^2 \quad (17)$$

which is the dispersion relation of transverse e.m. modes in plasma. The dispersion curve is shown below.



For e.g., the metallic lustre is attributed to the total reflection of visible light to which metals are not transparent. The reason being that the frequency of visible radiation is less than ω_p . The transparency of alkali metals in the ultraviolet region strongly indicates the theory that requires an e.m. radiation to have a certain minimum freq. or a wavelength not larger than a certain value to be able to pass through a plasma. The limit is defined in terms of ω_p or corresponding $\lambda_p (= 2\pi c / \omega_p)$.

