Advanced Mathematical Physics II

Representation of Group (3)

We ended last handout with mapping of operators on group element and under that we exhibited $\boxed{O_S O_T = O_{ST}}$ that there exist homomorphism.

Let us show "E" if its identity element then O_E is an identity operator.

$\boxed{EX = X}$ $\quad \therefore \quad O_E \psi(EX) = \psi(X)$ also $\boxed{O_E \psi(x) = \psi(x)}$

Also one may see $\quad O_E \cdot O_A = O_{E.A} = O_A \quad [\because E.A = A]$

$$O_A \cdot O_E = O_{A.E} = O_A$$

$\therefore$ O_E is an indentity operator.

Claim : $\boxed{O_{A^{-1}} = O_A^{-1}}$

Proof: $\quad O_{A.A^{-1}} = O_A \cdot O_{A^{-1}}$

$\boxed{O_E = O_A \cdot O_{A^{-1}}}$ $\quad \because$ O_E is an indentity operator

$\therefore \quad O_A \cdot O_A^{-1} = O_E \quad \because$ Inverse are unique. $\quad \therefore \boxed{O_{A^{-1}} = O_A^{-1}}$

Let us now demonstrate how can one use this mathematical technique to form a representation a group $\{E, I\}$ where "E" is identity element and "I" is inverse such that

If $\boxed{EX = X \; ; \quad IX = -X}$

$$O_E \psi(x) = \psi(E^{-1}x) = \psi(E.X) = \psi(X) = 1.\psi(x) + 0.\psi(-x)$$

$$O_I \psi(x) = \psi(I^{-1}x) = \psi(IX) = \psi(-X) = 0.\psi(x) + 1.\psi(-x)$$

If $\psi(x) = f_1$ and $\psi(-x) = f_2$ [f_1, f_2 are independent example e^x, e^{-x} are independent]

In the language of f_1 and f_2

$O_E f_1 = O_E \psi(x) = \psi(E^{-1} x) = \psi(x) = 1 \cdot f_1 + 0 \cdot f_2$

$O_E f_2 = O_E \psi(-x) = \psi(E^{-1} \cdot (-x)) = \psi(-x) = 0 \cdot f_1 + 1 \cdot f_2$

$O_I f_1 = O_I \psi(x) = \psi(I^{-1} \cdot x) = \psi(-x) = 0 \cdot f_1 + 1 \cdot f_2$

$O_I f_2 = O_I \psi(-x) = \psi(I^{-1} \cdot (-x)) = \psi(x) = 1 \cdot f_1 + 0 \cdot f_2$

If $O_R f_i = \sum_{j=1}^{2} f_j D_{ji}(R) \quad \forall\ i = 1, 2$

. Then $D(R)$ matrix represent O_R.

In ~~light~~ light of above definition.

$D(E) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ since

$O_E f_1 = f_1 D_{11}(E) + f_2 D_{21}(E)$

$O_E f_2 = f_1 D_{12}(E) + f_2 D_{22}(E)$

$\therefore D_{11}(E) = 1;\ D_{21}(E) = 0$

$D_{12}(E) = 0,\ D_{22}(E) = 1$

Similarly $D(I) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

* Mind it that its only one ~~for~~ of representation of very simple group G with two elements $\{E, I\}$ *

Let us take another example lets take $g_1(x)$ which is odd ~~function~~ function and $g_2(x)$ which is even function. [We may think $g_1(x) = \sin x$, $g_2(x) = \cos x$]

If we run the ~~analy~~ analysis again

$O_E g_1(x) = g_1(E^{-1} x) = g_1(x) = 1 \cdot g_1(x) + 0 \cdot g_2(x)$

$O_E g_2(x) = g_2(E^{-1} x) = g_2(x) = 0 \cdot g_1(x) + 1 \cdot g_2(x)$

$\therefore D(E) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$O_I g_1(x) = g_1(I^{-1} \cdot x) = g_1(I \cdot x) - g_1(-x) = -g_1(x) = -1 \cdot g_1 + 0 \cdot g_2$

$O_I g_2(x) = g_2(I^{-1} \cdot x) = g_2(I \cdot x) = g_2(-x) = g_2(x) = 0 g_1 + 1 \cdot g_2(x)$

$D(I) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

We may think of a plausible confusion $D(I)$ is represented by both $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ tied to different basis. To mark this difference in general we put subscript. $D^{(\mu)}(I)$ now $D^{(1)}(I)$ represent let us say representation with basis f_1, f_2 and $D^{(2)}(I)$ represents say with basis g_1, g_2.

$$\therefore \quad D^{(1)}(I) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad D^{(2)}(I) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

It may not appear as of now there is huge relevance since we already know they are operator tied to different basis vectors.

The importance is realise following substructure that is present g_1 and g_2 forms an independent one dimensional ~~represented~~ representation to say if we have only one function say

$\cancel{D^{(3)}(E) =}$ $g_1(x)$ such that $g_1(x)$ is odd.

Then $D^{(3)}(E) = (1)$ and $D^{(3)}(I) = (-1)$

we mark (3) to say its different representation.

only for. $g_2(x)$

$$D^{(4)}(E) = (1) \text{ and } D^{(4)}(I) = 1$$

We may think

$$D^{(2)}(I) = \begin{pmatrix} D^{(3)}(I) & 0 \\ 0 & D^{(4)}(I) \end{pmatrix}$$

That is existence of lower dimensional representation in higher ~~dimen~~ dimensional representation. We shall explore more on this in next handout.