Advanced Mathematical Physics II

Representation of Group (2)

Let there be a group with elements say $G = \{E, A, B, \dots\}$ such that "E" is indentity element and "G" is discrete and finite group.

If we can map "G" to set of operators $D(G)$ defined on vector space "L" homomorphically then $D(G)$ is said to be representation of group G.

If $\exists$ map $G \longrightarrow D(G)$. such that

$D(R \cdot S) = D(R) \cdot D(S)$. Then $D(G)$ is "one of" represenation of group "G".

Claim: "E" element will map identity operator of $D(G)$ which leaves all vectors belonging to "L" as invariant.

Proof: $D(R \cdot E) = D(R) \cdot D(E)$ [$\because$ its homomorphic map]

$D(R) = D(R) \cdot D(E)$ [$R \cdot E = R$; $\because$ "E" is indentity element]

Also $D(E \cdot R) = D(E) \cdot D(R) \Rightarrow D(R) = D(E) \cdot D(R)$

$\therefore \forall\, R \in G$.

$\Rightarrow$ This show $D(E)$ is indentity element of $D(G)$.

Claim: If 'R' is mapped to $D(R)$ then (R^{-1}) will be mapped to $D^{-1}(R)$

Proof: $D(R \cdot R^{-1}) = D(R) \cdot D(R^{-1})$ [Homomorphism]

$D(E) = D(R) \cdot D(R^{-1})$ [$\because R \cdot R^{-1} = E$ $R \in G, R^{-1} \in G$]

Also $D(R)\, D^{-1}(R) = D(E)$ [$D(G)$ is group if $D(R) \in D(G)$ Then $D^{-1}(R) \in D(G)$]

$\therefore \boxed{D(R^{-1}) = D^{-1}(R)}$

Let us take elements of "G" and map them to operators acting on vector space of functions. In order to ~~facilitate~~ facilitate this mathematically we define following

say $T \in G$ and $x \in$ "L" (vector space such that

$\boxed{Tx = x'}$ we map "T" to operator O_T such that

$$\boxed{O_T \psi(x) = \psi'(x)}$$

If "T" is symmetry operator then $\boxed{\begin{aligned} O_T \psi(x') &= \psi'(x') \\ &= \psi(x) \end{aligned}}$

$\therefore \quad \psi'(x') = \psi(x)$ [It means change $x \to x'$ under T and O_T changes ~~ψ~~ $\psi(x)$ to $\psi'(x)$ in such way that if we do change simultaneously we ~~not~~ recover the original function Back]

$$\because \quad O_T \psi(Tx) = O_T \psi(x') = \psi'(x') = \psi(x)$$

$$\therefore \quad O_T \psi(Tx) = \psi(x)$$ [This relation is true for all $x \in L$]

Then this should be true of $T^{-1}x$ also.

$$O_T \psi(T T^{-1} x) = \psi(T^{-1} x)$$

$$\therefore \quad \boxed{O_T \psi(x) = \psi(T^{-1} x)}$$

This defining definition of symmetry operator.

Example: Let "T" ~~to~~ be a translation operator $Tx = x' = x + a$ i.e. "x" is increased

$\therefore$ T translates $x = x' = x + a$ by "a"; The action T^{-1} will be obviously $T^{-1}x = x - a$ it reverse translates so that vector is back.

$$O_T \psi(x) = \psi(T^{-1}x) = \psi(x-a)$$

Thus we have found action of O_T on $\psi(x)$.

Let us define $Sx' = x''$

$$O_s \psi'(x'') = \psi''(x'') \qquad [\psi''(x'') = \psi'(x') = \psi(x)]$$

$$O_s O_T \psi(x'') = \psi(x)$$

$$O_s O_T \psi(Sx') = \psi(x)$$

$$O_s O_T \psi(STx) = \psi(x) \qquad [\text{Replace } x \text{ by } T^{-1}S^{-1}x]$$

$$O_s O_T \psi(STT^{-1}S^{-1}x) = \psi(T^{-1}S^{-1}x)$$

$$O_s O_T \psi(x) = \psi(T^{-1}S^{-1}x)$$

Consider $O_{ST}\psi(x) = \psi((ST)^{-1}x)$

$$O_{ST}\psi(x) = \psi(T^{-1}S^{-1}x)$$

$\left[\because (ST)^{-1}.\,ST = E \quad \because \text{inverse is } ~~can~~ \text{ unique}\right.$

$\therefore T^{-1}S^{-1}.\,S.T = T^{-1}.E.T = T^{-1}.T = E$

$\left.\therefore (ST)^{-1} = T^{-1}S^{-1}\right]$

$$\therefore \boxed{O_s O_T = O_{ST}}$$

Homomorphic map.