Advanced Mathematical Physics (II)

Factor group and complete group

If H is normal subgroup of G then coset (left or right) forms a group. This group is known as factor group. Generally denoted by G/H

Proof: Let H has elements $= \{e, h_1, h_2 \ldots h_{m-1}\}$
and G has elements $= \{e, g_1, g_2 \ldots g_{n-1}\}$.

$g \cdot H = H \cdot g$ [For a normal subgroup left and right coset are same]

Let us say coset of H are $e \cdot H$, $g_{j_1} \cdot H$, $g_{j_2} \cdot H$, ... and so on.

Claim: The elements $e \cdot H$, $g_{j_1} \cdot H$, $g_{j_2} \cdot H$, ..., $g_{j_{r-1}} \cdot H$ form a group.

Definition of Binary operation for such elements

$$F/H = \{e \cdot H, g_{j_1} \cdot H, g_{j_2} \cdot H, \ldots g_{j_{r-1}} \cdot H\}$$

If $g_{j_k} \cdot H \in F/H$ and $g_{j_l} \cdot H \in F/H$

Then $(g_{j_k} \cdot H) * (g_{j_l} \cdot H) := (g_{j_k} \cdot g_{j_l}) \cdot H$

By definition the binary operation

Let us prove under above given binary operation F/H forms a group.

Consider $g_{jk} \forall G$

$$(e \cdot H) * (g_{jk} \cdot H) = (e \cdot g_{jk}) \cdot H = g_{jk} \cdot H$$

also; $(g_{jk} \cdot H) * (e \cdot H) = (g_{jk} \cdot e) \cdot H = g_{jk} \cdot H$

$\therefore (e \cdot H) * (g_{jk} \cdot H) = (g_{jk} \cdot H) * (e \cdot H) = g_{jk} \cdot H$

$\therefore$ $e \cdot H$ is identity element of F/H

If $g_{jk} \cdot H \in F/H$ and $g_{jl} \cdot H \in F/H$

$$(g_{jk} \cdot H) * (g_{jl} \cdot H) = (g_{jk} \cdot g_{jl}) \cdot H$$

The ~~one~~ elements of $g_{jk} \cdot H$ ~~is~~ is $g_{jk} \cdot h_r$ (say)

The one element of $g_{jl} \cdot H$ ~~is~~ is $g_{jl} \cdot h_m$ (say)

Consider; $g_{jk} \cdot h_r \cdot g_{jl} \cdot h_m = g_{jk} \cdot g_{jl} \cdot h_n \cdot h_m$ (say $h_r \cdot g_{jl} = g_{jl} \cdot h_n$) $\because$ "H" is normal subgroup.

$= g_{jk} \cdot g_{jl} \cdot h_p$ (say)

is form $= (g_{jk} \cdot g_{jl}) H$

Also since $g_{jk} \cdot H \cap g_{jl} \cdot H = \phi$ (empty)

$\therefore (g_{jk} \cdot H) * (g_{jl} \cdot H) = (g_{jk} \cdot g_{jl}) \cdot H$ is closed.

$$(g_{jk} \cdot H) * (g_{jk}^{-1} \cdot H) = (g_{jk} \cdot g_{jk}^{-1}) \cdot H = e \cdot H$$

If $g_{jk} \in G$; $g_{jk} \cdot H \in F/H$; $g_{jk}^{-1} \in G$; $g_{jk}^{-1} \cdot H \in F/H$

Thus ~~iden~~ inverse ~~exsit~~ exist.

Automorphism

If a homomorphism maps a group G to itself it is called as automorphism

$$A : G \to G.$$

Inner Automorphism

One can always define a conjugacy map which shall be homomorphism of a group on group itself.

$$G = \{e, g_1, \ldots \; g_{n-1}\}$$

Consider following map

$$I : G \to G \quad : \quad I(g_i) = g_k^{-1} g_i g_k \quad \forall \; g_i \in G.$$

$$I(g_i \cdot g_j) = g_k^{-1} \cdot g_i \cdot g_j \cdot g_k$$

$$= g_k^{-1} \cdot g_i \cdot g_k \cdot g_k^{-1} \cdot g_j \cdot g_k$$

$$= I(g_i) \cdot I(g_j).$$

This map is called as inner automorphism.

Complete Group

A group is said to be complete if every automorphism is inner automorphism and $Z(G)$ is trivial (that is it has only identity element)