Advanced Mathematical Physics II

Conjugate Elements, Normal Subgroups and Centre of Group [Part 2]

Consider symmetry of square:

The subgroups are $\{E\}$; $\{E, C_4^2\}$, $\{E, M_x\}$, $\{E, M_y\}$
$\{E, \sigma_u\}$; $\{E, \sigma_v\}$; $\{E, C_4, C_4^2, C_4^3\}$; $\{E, C_4^2, M_x, M_y\}$
$\{E, C_4^2, \sigma_u, \sigma_v\}$

Left coset of $\{E, C_4^2\}$ are [choose other than E, C_4^2]

$C_4 \cdot \{E, C_4^2\} = \{C_4, C_4^3\}$ [Choose other than E, C_4^2, C_4, C_4^3]

$\sigma_u \{E, C_4^2\} = \{\sigma_u, \sigma_v\}$ [Choose other than $E, C_4, C_4^2, C_4^3, \sigma_u, \sigma_v$]

$M_x \{E, C_4^2\} = \{M_x, M_y\}$

$\therefore$ Left coset are $\{E, C_4^2\}$; $\{C_4, C_4^3\}$; $\{\sigma_u, \sigma_v\}$, $\{M_x, M_y\}$

Right coset of $\{E, C_4^2\}$

$\{E, C_4^2\} \cdot C_4 = \{C_4, C_4^3\}$

$\{E, C_4^2\} \cdot \sigma_u = \{\sigma_u, \sigma_v\}$

$\{E, C_4^2\} \cdot M_x = \{M_x, M_y\}$

Right coset are $\{E, C_4^2\}$, $\{C_4, C_4^3\}$; $\{\sigma_u, \sigma_v\}$, $\{M_x, M_y\}$

$\therefore$ $\{E, C_4^2\}$ is Normal subgroup $\because$ Right coset is same as left coset.

$\therefore$ One can write if H is normal subgroup then

$g_i \cdot H = H \cdot g_i$ [✱ Warning ordering does not matter ✱]

consider another fact the collection $\{E, C_4^2\}$ was also known as centre of group!!

Consider subgroup $H = \{E, C_4, C_4^2, C_4^3\}$
choose an element out side "H" say σ_u.

$$\sigma_u \cdot H = \sigma_u \cdot \{E, C_4, C_4^2, C_4^3\}$$
$$= \{\sigma_u, M_x, \sigma_v, M_y\}$$

$\therefore$ left coset of $\{E, C_4, C_4^2, C_4^3\}$ are $\{E, C_4, C_4^2, C_4^3\}$ and $\{\sigma_u, M_x, \sigma_v, M_y\}$

Right coset of $H = \{E, C_4, C_4^2, C_4^3\}$

$$\{E, C_4, C_4^2, C_4^3\} \cdot \sigma_u = \{\sigma_u, M_y, \sigma_v, M_x\}$$

Right coset of $\{E, C_4, C_4^2, C_4^3\}$ are $\{E, C_4, C_4^2, C_4^3\}$ and $\{\sigma_u, M_y, \sigma_v, M_x\}$

left coset = Right coset since ordering does not matter

$\therefore$ $\{\sigma_u, M_x, \sigma_u, M_y\}$ is same as $\{\sigma_u, M_y, \sigma_v, M_x\}$

$\therefore$ $g_i H = H g_i$ should be considered as $g_i \cdot h_k = h_r \cdot g_i$ it may not be in given order.

$\{E, C_4, C_4^2, C_4^3\}$ is NOT THE CENTRE OF GROUP!!

Self check Excercise

Find all cosets (left and right) of all other subgroups and identify which one of them is normal subgroup. in case of operation of symmetry of square.

Claim: Centre of group denoted by $Z(G)$ is normal subgroup.

Proof: $Z(G) = \{g_i : g_i \cdot g_k = g_k \cdot g_i \;\forall\; g_k \in G\}$

Consider $Z(G) = \{E, g_1^{(z)}, g_2^{(z)}, g_3^{(z)} \cdots g_m^{(z)}\}$

such that $g_i^{(z)} \cdot g_k = g_k \cdot g_i^{(z)} \quad \forall\ g_k \in G$ and $\forall\ g_i^{(z)} \in Z(G)$

$\therefore$

$$g_k \cdot Z(G) = \{g_k \cdot E,\ g_k \cdot g_1^{(z)},\ g_k \cdot g_2^{(z)}, \cdots ; g_k \cdot g_m^{(z)}\}$$

$$= \{E \cdot g_k,\ g_1^{(z)} \cdot g_k,\ g_2^{(z)} \cdot g_k \cdots ; g_m^{(z)} \cdot g_k\}$$

$$[\because g_i^{(z)} \cdot g_k = g_k \cdot g_i^{(z)}]$$

$$= Z(G) \cdot g_k$$

$\therefore$ $Z(G)$ is normal subgroup.

Self check exercise: show $Z(G)$ forms a group.

Question 4 Find normal subgroup of symmetry of triangle and S_3.

Question 5 Find centre of group for symmetry of triangle and S_3.

GOOGLE FORM WILL BE UPLOADED BY SUNDAY EVENING ON WEBPAGE FOR UPLOADING SOLUTIONS OF QUESTIONS. STUDENTS SHOULD FEEL FREE TO ASK THEIR DOUBTS. THERE WILL BE QUESTION ANSWERS SECTION FOR QUERY TOO POSTED BY SUNDAY EVENING.