Limits (ϵ-δ definition)

Definition:- Let $f(x)$ be a function defined for all x in some open interval containing the number 'c' except possibly at c. We write $\lim\limits_{x \to c} f(x) = L$

if given $\epsilon > 0$, $\exists\ \delta > 0$ s.t $0 < |x-c| < \delta \Rightarrow |f(x) - L| < \epsilon$.

Remarks:- (1) δ depends on ϵ and is not unique.

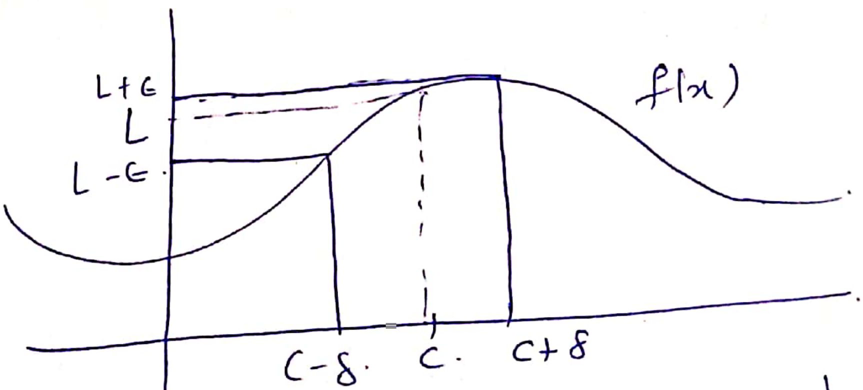
(2) The inequality $0 < |x-c|$ is equivalent to saying that $x \neq C$

(3) If L is the limit of $f(x)$ at $x = C$, then we say that $f(x)$ converges to L at c.

If the limit of $f(x)$ does not exist, then we say that $f(x)$ diverges at c.

(4) Limit of a function is always unique.

Geometric meaning



A function f is said to tend to a numbe L as "$x \to c$" if ~~f~~ given $\epsilon > 0$ s.t for $(L-\epsilon, L+\epsilon)$ then $\exists\ \delta > 0$ s.t for $x \in (c-\delta, c+\delta)$, $x \neq C$,

$f(x) \in (L-\epsilon, L+\epsilon)$

As ϵ gets smaller and smaller "$f(x)$ becomes closer and closer to L".

Example(1) Use $(\epsilon - \delta)$ definition to show that

$$\lim_{x \to 3} (3x-7) = 2.$$

Soln: We have
$f(x) = 3x - 7$, $L = 2$, $c = 3$.

Let $\epsilon > 0$ be given. Then

To find: $\delta > 0$ s.t. $|f(x) - L| < \epsilon$ if $|x-c| < \delta$

$$|f(x) - L| = |3x - 7 - 2|$$
$$= |3x - 9|$$
$$= 3|x - 3|.$$

Now $|f(x) - L| < \epsilon$ if $3|x-3| < \epsilon$
ie $|x-3| < \epsilon/3$.

So we choose $\delta = \epsilon/3$. Then
$|f(x) - L| < \epsilon$, if $0 < |x-3| < \epsilon/3$.

Since $\epsilon > 0$ is arbitrary, we deduce that
$\lim_{x \to 3} (3x - 7) = 2$.

② Prove that $\lim_{x \to 2} (3x - 5) = 1$. $\left(\lim_{x \to 1} (5x - 3) = 2 \quad \delta = \epsilon/5 \right)$

$|f(x) - L| = |3x - 5 - 1| = |3x - 6| = 3|x-2| < \epsilon$ if $|x-2| < \epsilon/3$

$(\delta = \epsilon/3)$

Eg: Finding δ in the definition of $\lim_{x \to c} f(x) = L$.

Find $\delta > 0$ s.t. $|f(x) - L| < \epsilon$, if $0 < |x-c| < \delta$.

(i) $\lim_{x \to -1} (7x + 5) = -2$, $\epsilon = 0.01$.

Soln: $f(x) = 7x + 5$, $L = -2$, $c = -1$, $\epsilon = 0.01$

$|f(x) - L| = |7x + 5 - (-2)| = |7x + 7| = 7|x+1|$.
$= 7|x - (-1)| < \epsilon$ whenever $|x-c| = |x-(-1)| < \epsilon/7$

$\delta = \frac{0.01}{7} = \frac{1}{700}$

Scanned with CamScanner

Eg: Prove that $\lim_{x \to 3} x^2 = 9$. ④

Soln: Let $\epsilon > 0$ be given.

To find: $\delta > 0$ s.t $|f(x) - L| < \epsilon$, if $0 < |x - c| < \delta$.

$f(x) = x^2$, $L = 9$, $c = 3$.

$|f(x) - L| = |x^2 - 9| = |x-3||x+3|$.

If we assume that $0 < \delta \leq 1$

We can consider: $0 < \delta \leq c$ for any $c > 0$.

Then.

$|x - 3| < 1$

$\Rightarrow -1 < x - 3 < 1$

$\Rightarrow 3 - 1 < x < 3 + 1$

$\Rightarrow 2 < x < 4$.

and thus.

$5 < x + 3 < 7$.

i.e, $|x+3| = x + 3 < 7$

Then. $|f(x) - L| = |x-3||x+3| < 7 \cdot |x - 3|$
$< 7\delta$
$< \epsilon$

if $7\delta < \epsilon$

i.e, $|f(x) - L| = |x+3||x-3| < \epsilon$ if, $0 < |x - 3| < \delta$.

for any $\delta > 0$ s.t $\delta \leq 1$ and, $\delta < \epsilon/7$.

Take $\delta = \min.\{1, \epsilon/7\}$.

δ | Note: ... be

Q Find $\delta > 0$ st

$$\lim_{x \to 4} x^2 = 16, \quad \epsilon = 0.001$$

$f(x) = x^2, \quad L = 16, \quad c = 4, \quad \epsilon = 0.001.$

~~δ-[illegible]~~.

$$|f(x) - L| = |x-4||x+4|$$

Assume $0 < \delta \leq 1$

$\Rightarrow |x-4| \leq 1$

$\Rightarrow 3 < x \leq 5$

$\Rightarrow 7 < x+4 < 9.$

$\Rightarrow |x+4| < 9$

$\Rightarrow |f(x) - L| < 9.|x-4| < \epsilon$

when. $9|x-4| < 9\epsilon/9 < \epsilon$.

ie, $\delta \leq \epsilon/9 = \dfrac{0.001}{9} = \dfrac{1}{9000}$.

Choose. $\delta = \min\{1, 1/9000\}$.

Eg: Prove that $\lim_{x \to 0^+} \sqrt{x} = 0$.

Soln $f(x) = \sqrt{x}$ $\therefore D(f) = \{x \in \mathbb{R} \mid x \geq 0\}$.

Thus we consider the limit as $x \to 0^+$.

Let $\epsilon > 0$ be given

To find: $\delta > 0$ s.t ~~when~~ $|\sqrt{x} - 0| < \epsilon$ if ~~x~~.

$0 < |x - 0| < \delta$

ie $0 < (x - 0) < \delta$ $(\because x \geq 0)$

Consider $|\sqrt{x} - 0| < \epsilon$

$\Rightarrow \sqrt{x} < \epsilon$

$\Rightarrow x < \epsilon^2.$

Take $\delta = \epsilon^2$.

D.

Q Find $\delta > 0$ s.t. (Not to be done)

$$\lim_{x \to 9} \sqrt{x} = 3, \quad \epsilon = 0.001$$

$$|f(x) - l| = |\sqrt{x} - 3| = \left|\frac{(\sqrt{x}-3)(\sqrt{x}+3)}{\sqrt{x}+3}\right|$$

1.

$$= \left|\frac{x-9}{\sqrt{x}+3}\right|.$$

$0 < \delta \leq 9$. Then

$$|x-9| < 9$$

$$0 < x < 18$$

$$0 < \sqrt{x} < \sqrt{18}$$

$$3 < \sqrt{x}+3 < \sqrt{18}+3.$$

$$\Rightarrow \frac{1}{\sqrt{x}+3} < \frac{1}{3}.$$

Then $|f(x) - L| = \left|\frac{x-9}{\sqrt{x}+3}\right| < \frac{1}{3}|x-9| < \epsilon$

whenever $|x-9| < 3\epsilon = 3(0.001)$
$= 0.003$.

$\therefore \delta = \min\{9, 0.003\} = 0.003$.

Q $\lim_{x \to 5} \sqrt{x-1} = 2$

① $|\sqrt{x-1} - 2| < 1$

$1 < x-1 < 9$

$\Rightarrow 2 < x < 10$

② $|x-5| < \delta$

$5-\delta < x < 5+\delta$

$3 \quad 7$

$2 \quad 5 \quad 10$

take $\delta = 3$.

Eg Prove that $\lim_{x \to -3} \frac{x^2-9}{x+3} = -6$.

Soln $f(x) = \frac{x^2-9}{x+3}$, $c = -3$, $l = -6$.

$$|f(x) - l| = \left|\frac{x^2-9}{x+3} - (-6)\right|$$

$$= \left|\frac{(x-3)(x+3)}{x+3} + 6\right| = |x+3| = |x-(-3)| < \epsilon$$

whenever $|x-(-3)| < \epsilon$.

ie Take $\delta = \epsilon$.

Since $\epsilon > 0$ is arb thus we get $\lim_{x \to -3} \frac{x^2-9}{x+3} = -6$.

7

Q ~~When $x \neq 1$, e.f.e.r~~

$$\lim_{x \to 1} f(x) = 1, \quad \text{if } f(x) = \begin{cases} x^2, & x \neq 1 \\ 2, & x = 1 \end{cases}$$

Soln $f(x) = x^2, \; x \neq 1$.

$l = 1. \quad c = 1.$

Let $\epsilon > 0$ be given.

$|f(x) - 1| = |x^2 - 1| = |x-1|\,|x+1|$.

Assume ~~$|x-c|$~~ $0 < \delta \le 1$

$\Rightarrow |x-1| < 1$

$\Rightarrow 2 < x+1 < 3$.

Then $|f(x) - l| = |x^2 - 1| = |x-1|\,|x+1| < 3|x-1|$
$< \epsilon$
if $|x-1| < \epsilon/3$

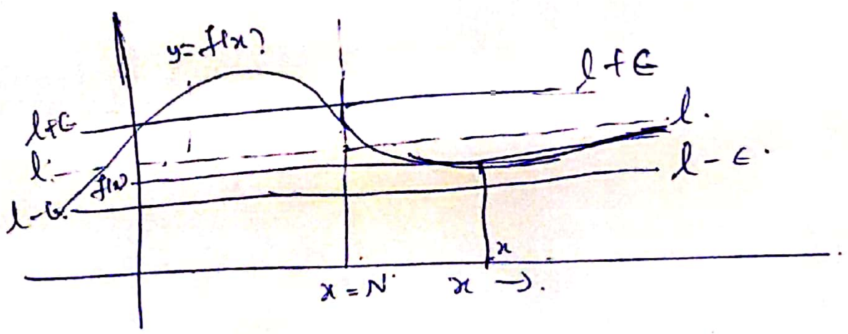
Choose $\delta = \min\{1, \frac{\epsilon}{3}\}$.

Limits at Infinity

Definition ① $\lim_{x \to +\infty} f(x) = l$. (ie, $f(x)$ gets closer and closer to l as x becomes suff. large)

Let $f(x)$ be defined for all x in some finite open interval extending in the positive x-direction. We will write $\lim_{x \to +\infty} f(x) = l$. if for given $\epsilon > 0$, $\exists$ a positive number N s.t

$$|f(x) - l| < \epsilon, \text{ if } x > N.$$



Eg: $\lim_{x\to\infty} \frac{1}{x} = 0$

Soln: Let $\epsilon > 0$ be given.

To find :- A +ve number N s.t $|f(x)-l| < \epsilon$, if $x > N$.

$|f(x)-l| = \left|\frac{1}{x} - 0\right| = \frac{1}{|x|} < \epsilon$ if $|x| > \frac{1}{\epsilon}$

$\Rightarrow x > \frac{1}{\epsilon}$.

Take $N = \frac{1}{\epsilon}$

Then $|f(x)-l| = \left|\frac{1}{x} - 0\right| < \epsilon$ whenever $x > N = \frac{1}{\epsilon}$ $\quad \frac{1}{|x|} < \frac{1}{x}$.

$\therefore \lim_{x\to+\infty} f(x) = 0$.

Eg: Find N if $\lim_{x\to+\infty} \frac{1}{x+2} = 0$, $\epsilon = 0.005$

$|f(x)-l| = \left|\frac{1}{x+2} - 0\right| < \epsilon$

$\Rightarrow \frac{1}{x+2} < \epsilon$

$\Rightarrow x+2 > \frac{1}{\epsilon}$

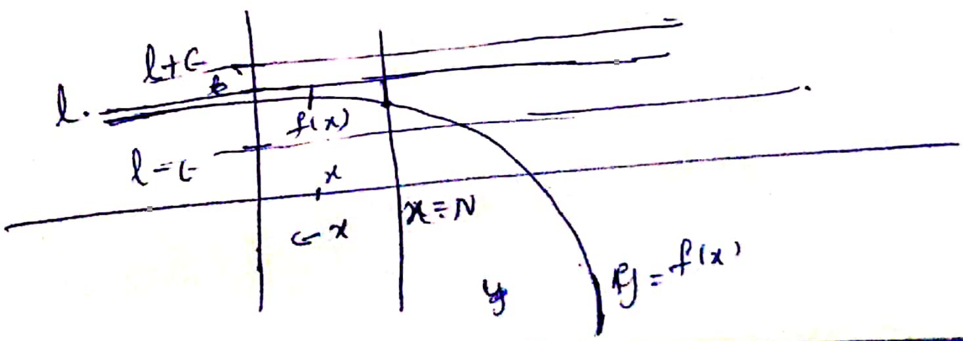
$\Rightarrow x > \frac{1}{\epsilon} - 2$

take $N = \frac{1}{\epsilon} - 2 = 198$.

(ie $f(x) \to l$ as x moves farther away from 0 on left)

Definition ② $\lim_{x\to-\infty} f(x) = l$.

Let $f(x)$ be defined for all x in some infinite open interval extending in the negative x direction. We will write $\lim_{x\to-\infty} f(x) = l$, if given $\epsilon > 0$, $\exists N < 0$ s.t $|f(x) - l| < \epsilon$, if $x < N$.



Q $\lim_{x \to -\infty} \frac{1}{x} = 0.$

Let $\epsilon > 0$ be given. To find: $N < 0$ - - .

$|f(x) - l| = |\frac{1}{x} - 0| < \epsilon$

$\Rightarrow -\frac{1}{x} < \epsilon$

$\Rightarrow 1/x > -\epsilon$

$\Rightarrow x < -\frac{1}{\epsilon}$

Take $N = -\frac{1}{\epsilon}$ - - -

Q Find N st
$\lim_{x \to -\infty} \frac{1}{x+2} = 0, \epsilon = 0.005$

$|f(x) - l| = |\frac{1}{x+2} - 0| < \epsilon$

$\Rightarrow \frac{1}{|x+2|} < \epsilon$

$\Rightarrow |x+2| > \frac{1}{\epsilon} = \frac{1}{0.005} = 200.$

$\boxed{|a| > a. \quad \frac{1}{|a|} < \frac{1}{a}. \quad \frac{1}{a} < \epsilon}$

$\Rightarrow -x-2 > 200.$

$\Rightarrow x < -202.$

Take $N = -202$.

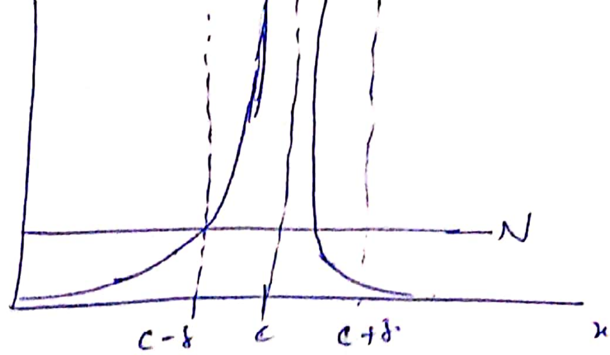
~~Defn~~ Infinite Limits:-

Definition (3) $\lim_{x \to c} f(x) = +\infty$

Let $f(x)$ be defined for all x in some open interval containing c, except ~~that the~~ possibly at c.
We will write, $\lim_{x \to c} f(x) = +\infty$ if for any +ve number M.

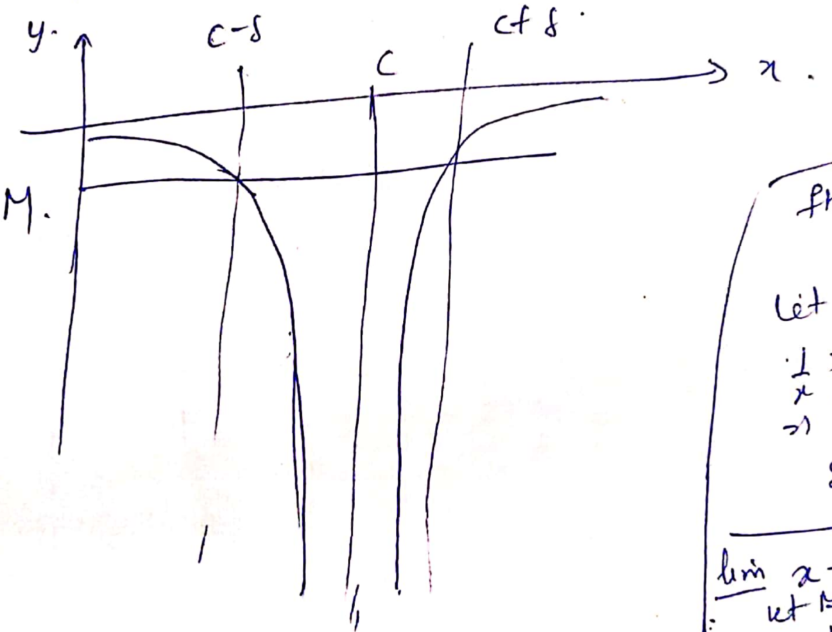
$\exists\ \delta > 0$ st $f(x) > M$, if $0 < |x-c| < \delta$.

Scanned with CamScanner



(4) $\lim_{x \to c} f(x) = -\infty$.

Let $f(x)$ be defined for all x in some open interval containing c except possible at c. We will write $\lim_{x \to c} f(x) = -\infty$ if for given any negative number M, $\exists\ \delta > 0$ s.t $f(x) < M$, if $0 < |x - c| < \delta$.



$f(x) = \frac{1}{x}$.
$c = 0^+$, $L = \infty$.
Let $M > 0$ $\quad x \to 0^+$ $\quad |x| = x$.
$\frac{1}{x} > M$.
$\Rightarrow x < \frac{1}{M}$.
$\delta = \frac{1}{M}$.

lim $x \to 0^-$. Let $M < 0$
$\frac{1}{x} < M$
$\Rightarrow x > \frac{1}{M}$.
$\Rightarrow -x < -\frac{1}{M}$.
$\delta = -1/M > 0$.

Eg: $\lim_{x \to 0} \frac{1}{x^2} = +\infty$.

$f(x) = 1/x^2$, $\quad c = 0$, $l = +\infty$.

Let $M > 0$ be given. To find $\delta > 0$ s.t $f(x) > M$ if $0 < |x - 0| < \delta$

$f(x) = \frac{1}{x^2} > M$

if. $x^2 < \frac{1}{M}$. $\Rightarrow |x| < \frac{1}{\sqrt{M}}$.

$\Rightarrow \delta = \frac{1}{\sqrt{M}}$.

Sequential Criterion for Limits:

Theorem :- Let f be defined in some interval ~~fee~~ $(c-\delta, c+\delta)$ except possibly at c. Then the following are equivalent.

(1) $\lim_{x \to c} f(x) = L$.

(2) For every sequence (x_n) with $0 < |x_n - c| < \delta$ for $n \in \mathbb{N}$, converging to c, the sequence $(f(x_n))$ converges to L.

Q(a) Evaluate the following Limits

① $\lim_{x \to 0} \frac{e^{1/x} - 1}{e^{1/x} + 1}$

We know, for $x \to 0^+$.

$1/x \to +\infty$.

$e^{1/x} \to +\infty$.

$\frac{1}{e^{1/x}} \to 0$. $\Rightarrow e^{-1/x} \to 0$.

Also for $x \to 0^-$, $\frac{1}{x} \to -\infty$. $\Rightarrow e^{1/x} \to 0$.

$\therefore$ LHL $= \lim_{x \to 0^-} \frac{e^{1/x} - d}{e^{1/x} + 1} = \frac{0-1}{0+1} = -1$

RHL $= \lim_{x \to 0^+} \frac{e^{1/x} - 1}{e^{1/x} + 1} = \lim_{x \to 0} \frac{1 - e^{-1/x}}{1 + e^{-1/x}} = 1$.

LHL $\neq$ RHL $\therefore$ Limit does not exist.

② $\lim_{x \to 0} \frac{x e^{1/x}}{e^{1/x} + 1}$

LHL. $\lim_{x \to 0^-} \frac{x e^{1/x}}{e^{1/x} + 1} = \frac{0.0}{0+1} = 0$

RHL $\lim_{x \to 0^+} \frac{x e^{1/x}}{e^{1/x} + 1} = \lim_{x \to 0^+} \frac{x}{1 + e^{-1/x}} = \frac{0}{1+0} = 0$

LHL = RHL

$\therefore \lim_{x \to 0} \frac{x e^{1/x}}{e^{1/x} + 1} = 0$.