Section 7.3 : The Minimal Polynomial

Definition : Let T be a linear operator on a FDVS. A polynomial p(t) is called a minimal polynomial of T if p(t) is a monic polynomial of least positive degree for which p(T) = O i.e. the zero operator.

Theorem 7.12 : Let p(t) be a minimal polynomial of a linear operator T on a FDVS V.

(a): For any polynomial g(t) if g(T) = O, then p(t) divides g(t). In particular p(t) divides the characteristic polynomial of T.

(b): The minimal polynomial of T is unique.

Proof (a) : Let g(t) be a polynomial for which g(T) = O.

By division algorithm, there exist polynomials q(t) and r(t) such that

$g(t) = q(t)\, p(t) + r(t)$ where $r(t) = 0$ or $\deg r(t) < \deg p(t)$.

Then $g(T) = q(T)\, p(T) + r(T)$

i.e. $O = q(T).\, O + r(T)$

which implies $r(T) = O$.

As $\deg r(t) < \deg p(t)$ and p(t) is the minimal polynomial of T, so r(t) must be zero.

Hence , p(t) divides g(t).

(b): Suppose that $p_1(t)$ and $p_2(t)$ are each minimal polynomials of T. Then $p_1(t)$ divides $p_2(t)$ by part (a). Since $p_1(t)$ and $p_2(t)$ have the same degree, we have $p_2(t) = c\, p_1(t)$ for some non zero scalar c . As $p_1(t)$ and $p_2(t)$ are monic, so c = 1. Hence $p_1(t) = p_2(t)$.

Definition : Let $A \in M_{n\times n}(F)$. The minimal polynomial p(t) of A is the monic polynomial of least positive degree for which $p(A) = O$, i.e. the zero matrix.

Theorem 7.13 : Let T be a linear operator on a FDVS V and let β be an ordered basis for V. Then the minimal polynomial of T is the same as the minimal polynomial of $[T]_\beta$.

Corollary : For any $A \in M_{n\times n}(F)$, the minimal polynomial of A is the same as the minimal polynomial of L_A .

Theorem 7.14 : Let T be a linear operator on a FDVS V and let p(t) be the minimal polynomial of T. A scalar λ is an eigenvalue of T if and only if p(λ) = 0. Hence the characteristic polynomial and the minimal polynomial of T have the same zeros.

Proof : Let f(t) be the characteristic polynomial of T. Since p(t) divides f(t), there exist polynomials q(t) and r(t) such that f(t) = q(t) p(t) .

If λ is a zero of p(t) then f(λ) = q(λ) p(λ) = q(λ).0 = 0. So λ is a zero of f(t) i.e. λ is an eigenvalue of T.

Conversely, suppose λ is an eigenvalue of T and let x ∈ V be an eigenvector corresponding to λ. Then T(x) = λ x and p(T)(x) = p(λ)x. As p(T) = O so 0 = O(x) = p(T)(x) = p(λ)x . Since x being an eigenvector is nonzero, it follows that p(λ) = 0 and so λ is a zero of p(t) also.

Corollary : Let T be a linear operator on a FDVS V with minimal polynomial p(t) and characteristic polynomial f(t). Suppose that f(t) factors as

$$f(t) = (\lambda_1 - t)^{n_1} \ (\lambda_2 - t)^{n_2} \ \ldots\ldots\ldots (\lambda_k - t)^{n_k}$$

where $\lambda_1, \lambda_2, \ldots\ldots, \lambda_k$ are the distinct eigenvalues of T. Then there exist integers $m_1, m_2, \ldots\ldots m_k$ such that $1 \le m_i \le n_i$ for all i = 1, 2,…..k, and

$$p(t) = (\lambda_1 - t)^{m_1} \ (\lambda_2 - t)^{m_2} \ \ldots\ldots\ldots (\lambda_k - t)^{m_k}$$

Example 1 : Compute the minimal Polynomial of the matrix

$$A = \begin{pmatrix} 3 & -1 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 2 \end{pmatrix}$$

Sol. : The characteristic polynomial of A is det (A – tI) i.e. $\det\begin{pmatrix} 3-t & -1 & 0 \\ 0 & 2-t & 0 \\ 1 & -1 & 2-t \end{pmatrix}$ i.e. $f(t) = -(t-2)^2(t-3)$

As the minimal polynomial p(t) divides f(t) and they have the same zeros, so possibilities for p(t) are

$(t-2)(t-3)$ or $(t-2)^2(t-3)$

If we let $p(t) = (t-2)(t-3)$ then $p(A) = (A-2I)(A-3I) = A^2 - 5A + 6I = O$.

Thus p(t) is a polynomial of least degree satisfying p(A) = O, so the minimal polynomial

of A is $p(t) = (t-2)(t-3)$.

Example 2 : Let T be the linear operator on R^2 defined by T(a, b) = (2a+5b, 6a+b). Compute minimal polynomial of T.

Sol. : Let β be standard ordered basis for R^2. Then $[T]_\beta = \begin{pmatrix} 2 & 5 \\ 6 & 1 \end{pmatrix}$. The characteristic polynomial of T is $f(t) = \begin{vmatrix} 2-t & 5 \\ 6 & 1-t \end{vmatrix} = (t-7)(t+4)$.

As characteristic polynomial and minimal polynomial have the same zeros,

therefore the minimal polynomial is also $(t-7)(t+4)$.

Example 3. : Let D be the linear operator on $P_2(R)$ defined by D(g(x)) = g′(x) .

Compute the minimal polynomial of D.

Sol. : Let β = { 1, x, x^2 } be standard ordered basis for $P_2(R)$. Then $[D]_\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$.

The characteristic polynomial of D is $-t^3$.

The possibilities for minimal polynomial are t, t^2, t^3 .

As $D \neq O$ and $D^2 \neq O$ but $D^3 = O$. Therefore the minimal polynomial of D is t^3.

Theorem 7.15 : Let T be a linear operator on an n – dimensional vector space V such that V is a T – cyclic subspace of itself. Then the characteristic polynomial f(t) and the minimal polynomial $p(t)$ have the same degree, and hence $f(t) = (-1)^n p(t)$.

Proof : Since V is a T – cyclic space , $\exists$ a nonzero $x \in V$ such that

$\beta = \{ x, T(x), \ldots, T^{n-1}(x) \}$ is a basis for V . (By Theorem 5.22).

Let $g(t) = a_0 + a_1 t + \ldots + a_k t^k$ be a polynomial of degree $k < n$. Then $a_k \neq 0$

and $g(T)(x) = a_0 x + a_1 T(x) + \ldots + a_k T^k(x)$.

Observe that $g(T)(x)$ is a linear combination of elements of β such that $a_k \neq 0$.

Then $g(T)(x) \neq 0$. (Since β is linearly independent) . Hence $g(T) \neq O$.

Therefore, the minimal polynomial of T has degree n which is also the degree of the characteristic polynomial. Hence, $f(t) = (-1)^n p(t)$.

Theorem 7.16 : Let T be a linear operator on a FDVS V. Then T is diagonalizable if and only if the minimal polynomial of T is of the form

$$p(t) = (t - \lambda_1) (t - \lambda_2) \ldots\ldots (t - \lambda_k)$$

where λ_1 , λ_2 ,…… λ_k are the distinct eigenvalues of T .

Proof : Not to be done.

Example 4 : Determine all matrices $A \in M_{2\times2}(R)$ for which $A^2 - 3A + 2I = O$.

Sol. : Let $g(t) = t^2 - 3t + 2 = (t - 1) (t - 2)$.

As $g(A) = A^2 - 3A + 2I = O$, the minimal polynomial $p(t)$ of A divides $g(t)$.

Hence the possibilities for $p(t)$ are $t - 1$ or $t - 2$ or $(t - 1) (t - 2)$.

If $p(t) = t - 1$ then $A = I$ satisfies the given condition.

If $p(t) = t - 2$ then $A = 2I$ satisfies the given condition.

If $p(t) = (t - 1) (t - 2)$ then A is diagonalizable with eigenvalues 1 and 2 and hence A is similar to $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

Example 5 : Let $A \in M_{n\times n}(R)$ satisfy $A^3 = A$. Show that A is diagonalizable .

Sol. : Let $g(t) = t^3 - t = t(t-1)(t+1)$. Then $g(A) = O$ and hence the minimal polynomial p(t) of A divides g(t). Hence the possibilities for p(t) are t, $t-1$ or $t+1$ or $t(t-1)$ or $t(t+1)$ or $(t-1)(t+1)$ or $t(t-1)(t+1)$. In each of these possibilities p(t) has no repeated factors . Thus A is diagonalizable (by Theorem 7.16).

Example 6 : In example 3 we have seen that the minimal polynomial of differential operator D is t^3. Thus it has repeated zeros. So by Theorem 7.16, D is not diagonalizable.

Assignment : Do questions 5, 8 and 10 from exercise 7.3 .