6.3. The Adjoint of a L.O.

Theorem 6.8: Let V be a FD IPS over F & let $g: V \to F$ be a LT, then $\exists$ a unique vector $y \in V$, such that $g(x) = \langle x, y \rangle \ \forall \ x \in V$

Pf: Let $\beta = \{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis for V & let $y = \sum_{i=1}^{n} \overline{g(v_i)}\, v_i$. Then $y \in V$

Define $h: V \to F$ by $h(x) = \langle x, y \rangle \ \forall \ x \in V$.

Let $x_1, x_2 \in V$, $c \in F$, then

$$h(cx_1 + x_2) = \langle cx_1 + x_2, y \rangle = c\langle x_1, y \rangle + \langle x_2, y \rangle$$
$$= c\,h(x_1) + h(x_2)$$

$\Rightarrow h$ is a LT from V to F.

Also for $1 \le j \le n$, we have

$$h(v_j) = \langle v_j, y \rangle = \langle v_j, \sum_{i=1}^{n} \overline{g(v_i)}\, v_i \rangle = \sum_{i=1}^{n} g(v_i)\langle v_j, v_i \rangle$$
$$= \sum_{i=1}^{n} g(v_i)\,\delta_{ji} = g(v_j)$$

Let $x \in V$, then $x = \sum a_i v_i$, where $a_i \in F \ \forall i$

Then
$$h(x) = h\left(\sum a_i v_i\right) = \sum a_i h(v_i) = \sum_i a_i g(v_i)$$
$$= g\left(\sum_i a_i v_i\right) = g(x)$$

$\Rightarrow h(x) = g(x) \ \forall \ x \in V$ & so $h = g$ on V

To show that y is unique, suppose that

$g(x) = \langle x, y' \rangle \ \forall \ x$

Then $\langle x, y \rangle = \langle x, y' \rangle \ \forall \ x \in V$

$\Rightarrow \langle x, y - y' \rangle = 0 \ \forall \ x \in V$

In particular, $\langle y - y', y - y' \rangle = 0 \Rightarrow \|y - y'\|^2 = 0$

$\Rightarrow y = y'$

Ex.1 Define $g: \mathbb{R}^2 \to \mathbb{R}$ by $g(a_1, a_2) = 2a_1 + a_2$;

Clearly g is a L.T. $\because$ $g((a_1, a_2) + (b_1, b_2)) = g(a_1+b_1, a_2+b_2)$
$= 2(a_1+b_1) + (a_2+b_2)$
$= 2a_1 + a_2 + 2b_1 + b_2$
$= g(a_1, a_2) + g(b_1, b_2)$

& $g(c(a_1, a_2)) = g(ca_1, ca_2)$
$= 2(ca_1) + ca_2$
$= c(2a_1 + a_2) = c\,g(a_1, a_2)$

Let $\beta = \{e_1, e_2\}$

β is an orthonormal basis for $\mathbb{R}^2$

Let $y = g(e_1)e_1 + g(e_2)e_2$
$= 2e_1 + e_2 = (2, 1)$

Then $g(a_1, a_2) = \langle (a_1, a_2), y \rangle = \langle (a_1, a_2), (2, 1) \rangle = 2a_1 + a_2$

Theorem 6.9: Let V be a FD IPS & let T be a LO on V. Then $\exists$ a unique function $T^*: V \to V$ such that $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$ $\forall x, y \in V$. Furthermore, T^* is linear.

Proof: Let $y \in V$. Define $g: V \to F$ by,
$g(x) = \langle T(x), y \rangle$ $\forall x \in V$.

Then g is a LT from V into F.

Let $x_1, x_2 \in V$ and $c \in F$. Then

$g(cx_1 + x_2) = \langle T(cx_1 + x_2), y \rangle = \langle c\,T(x_1) + T(x_2), y \rangle$
$= c\langle T(x_1), y \rangle + \langle T(x_2), y \rangle = c\,g(x_1) + g(x_2)$

Also by Thm. 6.8, $\exists$ a unique vector $y' \in V$ such that $g(x) = \langle x, y' \rangle$ ie $\langle T(x), y \rangle = \langle x, y' \rangle$ $\forall x \in V$

Define $T^*: V \to V$ by $T^*(y) = y'$

Then $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$ $\forall x \in V$.

We shall now prove that T^* is linear.

Let $y_1, y_2 \in V$ & $c \in F$.

Then
$$\langle x, T^*(cy_1+y_2)\rangle = \langle T(x), cy_1+y_2\rangle$$
$$= \bar{c}\langle T(x), y_1\rangle + \langle T(x), y_2\rangle$$
$$= \bar{c}\langle x, T^*(y_1)\rangle + \langle x, T^*(y_2)\rangle$$
$$= \langle x, cT^*(y_1) + T^*(y_2)\rangle \quad \forall\, x\in V$$

$$\Rightarrow \quad T^*(cy_1+y_2) = c\,T^*(y_1) + T^*(y_2)$$

Finally, we shall prove that T^* is unique.
suppose that $U: V \to V$ is linear and that it satisfies
$\langle T(x), y\rangle = \langle x, U(y)\rangle \quad \forall\, x, y \in V.$
Then $\langle x, T^*(y)\rangle = \langle x, U(y)\rangle \quad \forall\, x, y \in V$ & so $T^* = U$

Remark (i) The LO T^* defined above is called the adjoint of the operator T.
Thus T^* is the unique operator on V satisfying,
$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle \quad \forall\, x, y\in V.$$

Remark (2). We also have
$$\langle x, T(y)\rangle = \overline{\langle T(y), x\rangle} = \overline{\langle y, T^*(x)\rangle} = \langle T^*(x), y\rangle$$
so $\langle x, T(y)\rangle = \langle T^*(x), y\rangle \quad \forall\, x, y\in V.$

Remark (3). For an infinite dimensional IPS, the existence of T^* is not guaranteed.

Theorem 6.10. Let V be a FD IPS & let β be an orthonormal basis for V. If T is a LO on V, then
$$[T^*]_\beta = [T]_\beta^*$$

Proof: Let $A = [T]_\beta$ & $B = [T^*]_\beta$ & $\beta = \{v_1, v_2, \ldots, v_n\}$

Then $B_{ij} = \langle T^*(v_j), v_i\rangle = \overline{\langle v_i, T^*(v_j)\rangle} = \overline{\langle T(v_i), v_j\rangle} = \overline{A_{ji}} = (A^*)_{ij}$

(By Corollary to Thm 6.5)

Hence, $B = A^*$.

Corollary: Let A be an $n\times n$ matrix, then $L_{A^*} = (L_A)^*$

Pf: If β is the SOB for F^n, then ~~by~~ we have $[L_A]_\beta = A$. Hence $[(L_A)^*]_\beta = [L_A]_\beta^* = A^* = [L_{A^*}]_\beta$ & so $(L_A)^* = L_{A^*}$.

Ex. 2. Let T be the LO on $\mathbb{C}^2$ defined by
$$T(a_1,a_2) = (2ia_1+3a_2,\ a_1-a_2)$$
find the adjoint of T.

Sol Let $\beta = \{e_1, e_2\}$ be SOB for $\mathbb{C}^2$ then $[T]_\beta = \begin{bmatrix} 2i & 3 \\ 1 & -1 \end{bmatrix}$

so $[T^*]_\beta = [T]_\beta^* = \begin{pmatrix} -2i & 1 \\ 3 & -1 \end{pmatrix}$

Hence $T^*(a_1,a_2) = (-2ia_1+a_2,\ 3a_1-a_2)$

Thm 6.11: Let V be an IPS & let T & U be the LO's on V. Then (a) $(T+U)^* = T^*+U^*$ (b) $(cT)^* = \bar{c}T^*$ for any $c\in F$ (c) $(TU)^* = U^*T^*$ (d) $T^{**} = U$ (e) $I^* = I$

Pf (a) Let $x, y \in V$, then
$$\langle x, (T+U)^*(y)\rangle = \langle (T+U)(x), y\rangle$$
$$= \langle T(x), y\rangle + \langle U(x), y\rangle$$
$$= \langle x, T^*(y)\rangle + \langle x, U^*(y)\rangle$$
$$= \langle x, (T^*+U^*)y\rangle$$
$\Rightarrow (T+U)^*(y) = (T^*+U^*)(y)\ \forall y\in V$

$\Rightarrow (T+U)^* = T^*+U^*$

(b) Let $x, y \in V$ then $\langle x, (cT^*)(y)\rangle = \langle (cT)(x), y\rangle$
$= \langle c\,T(x), y\rangle = c\langle T(x), y\rangle = \langle T(x), \bar{c}y\rangle = \langle x, T^*(\bar{c}y)\rangle$
$= \langle x, \bar{c}\,T^*(y)\rangle \Rightarrow (cT^*)(y) = (\bar{c}T^*)(y)\ \forall y\in V$

$\Rightarrow cT^* = \bar{c}T^*\ \forall\ c\in F.$

(c) Let $x, y \in V$,

$$\langle x, (TU)^*(y)\rangle = \langle (TU)(x), y\rangle = \langle T(U(x)), y\rangle$$
$$= \langle U(x), T^*(y)\rangle = \langle x, U^*(T^*(y))\rangle$$
$$= \langle x, (U^*T^*)(y)\rangle \quad \forall\, x, y \in V$$
$$\Rightarrow (TU)^*(y) = (U^*T^*)(y) \quad \forall\, y \in V \Rightarrow (TU)^* = U^*T^*$$

(d) $\langle x, T(y)\rangle = \langle T^*(x), y\rangle = \langle x, (T^*)^*(y)\rangle = \langle x, T^{**}(y)\rangle$

$$\Rightarrow T(y) = T^{**}(y) \quad \forall\, y \in V \Rightarrow T = T^{**}$$

(e) $\langle x, I(y)\rangle = \langle x, y\rangle = \langle I(x), y\rangle = \langle x, I^*(y)\rangle \quad \forall\, x \in V$

$$\Rightarrow I(y) = I^*(y) \quad \forall\, y \in V \Rightarrow I = I^*.$$

Corollary: Let A & B be $n \times n$ matrices then

(a) $(A+B)^* = A^* + B^*$ (b) $(cA)^* = \bar{c}A^* \quad \forall\, c \in F$

(c) $(AB)^* = B^*A^*$ (d) $A^{**} = A$ (e) $I^* = I$

Pf: (a) We have $L_{(A+B)^*} = (L_{(A+B)})^* = (L_A + L_B)^* = L_A^* + L_B^*$

$$= L_{A^*} + L_{B^*} = L_{A^*+B^*}$$
$$\Rightarrow (A+B)^* = A^* + B^*$$

(b) $(L_{cA})^* = (cL_A)^* = \bar{c}L_A^* = \bar{c}L_{A^*} = L_{\bar{c}A^*}$ for any $c \in F$

(c) $L_{(AB)^*} = (L_{AB})^* = (L_A L_B)^* = L_B^* L_A^* = L_{B^*}L_{A^*} = L_{B^*A^*}$

$$\Rightarrow (AB)^* = B^*A^*$$

(d) $L_{A^{**}} = (L_{A^*})^* = ((L_A)^*)^* = L_A^{**} = L_A$

$$\Rightarrow A^{**} = A$$

(e) $L_{I_n^*} = (L_{I_n})^* = I = L_{I_n} \Rightarrow I_n^* = I_n$

Least Square approximations

Consider some data $(t_1, y_1), (t_2, y_2), \ldots, (t_m, y_m)$ plotted as points in the plane, From this plat it appears that $\exists$ an essentially linear relationship between y & t say $y = ct + d$. We would like to find the constants c & d so that the line $y = ct + d$ represents the best possible fit to the data collected. One such estimate of fit is to calculate the error E that represents the sum of the squares of the vertical distances from the points to the line; that is

$$E = \sum_{i=1}^{m} (y_i - ct_i - d)^2$$

Thus the problem is reduced to finding the constants c and d that minimize E.

Remark: The line $y = ct + d$ is called the least square line

Remark: If we let $A = \begin{bmatrix} t_1 & 1 \\ t_2 & 1 \\ \vdots & \\ t_m & 1 \end{bmatrix}$, $x = \begin{pmatrix} c \\ d \end{pmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$

then $E = \|y - Ax\|^2$

The problem is to find an explicit vector $x_0 \in F^n$ that minimizes E. ie given an $m \times n$ matrix A, we find $x_0 \in F^n$ such that
$\|y - Ax_0\| \leq \|y - Ax\|$ $\forall$ vectors $x \in F^n$.

Remark: This method not only allows us to find the linear function that best fits the data, but also, for any positive integer n, the best fit using a polynomial of degree at most n.

First we need some notation and two simple lemmas (without proof), Corollary & Thm 6.12

(i) For $x, y \in F^n$, $\langle x, y \rangle_n$ denote standard IP of x & y in F^n.

Lemma 1 Let $A \in M_{m\times n}(F)$, $x \in F^n$ & $y \in F^m$. Then

$$\langle Ax, y \rangle_m = \langle x, A^* y \rangle_n$$

Lemma 2. Let $A \in M_{m\times n}(F)$. Then $\text{rank}(A^*A) = \text{rank}\, A$

Corollary: If A is an $m\times n$ matrix such that $\text{rank}(A) = n$ then A^*A is invertible.

Theorem 6.12: Let $A \in M_{m\times n}(F)$ and $y \in F^m$. Then $\exists$ $x_0 \in F^n$ such that $(A^*A)x_0 = A^*y$ and $\|Ax_0 - y\| \leq \|Ax - y\|$ $\forall$ $x \in F^n$. Furthermore, if $\text{rank}\, A = n$, then $x_0 = (A^*A)^{-1}A^*y$.

Ex. Let's suppose that the data collected are $(1,2), (2,3), (3,5)$ & $(4,7)$. Then

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \quad \& \quad y = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix}$$

hence $A^*A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} = \begin{pmatrix} 30 & 10 \\ 10 & 4 \end{pmatrix}$

Thus $(A^*A)^{-1} = \frac{1}{20} \begin{pmatrix} 4 & -10 \\ -10 & 30 \end{pmatrix}$

$\therefore \begin{pmatrix} c \\ d \end{pmatrix} = x_0 = \frac{1}{20} \begin{pmatrix} 4 & -10 \\ -10 & 30 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 1.7 \\ 0 \end{bmatrix}$

Thus the line $y = 1.7t$ is the least square line. The error E is given by

$E = \|Ax_0 - y\|^2 = \langle Ax_0 - y, Ax_0 - y \rangle = .09 + 0.16 + .01 + .04$
$= 0.3$

$Ax_0 - y = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1.7 \\ 0 \end{bmatrix} - y = \begin{bmatrix} 1.7 \\ 3.4 \\ 5.1 \\ 6.8 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} -0.3 \\ .4 \\ .1 \\ -.2 \end{bmatrix}$

Remark: The above method may also be applied for fitting a quadratic polynomial.
suppose one has to fit the poly. $y = ct^2 + dt + e$
the appropriate model is

$$x = \begin{bmatrix} c \\ d \\ e \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \quad \& \quad A = \begin{pmatrix} t_1^2 & t_1 & 1 \\ \vdots & \vdots & \vdots \\ t_m^2 & t_m & 1 \end{pmatrix}$$

Q.20
Sol (i) a linear function
The data is $(-3, 9)$ $(-2, 6)$, $(0, 2)$ $(1, 1)$

$\therefore A = \begin{bmatrix} -3 & 1 \\ -2 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad y = \begin{bmatrix} 9 \\ 6 \\ 2 \\ 1 \end{bmatrix} \quad A^*A = \begin{bmatrix} -3 & -2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ -2 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 14 & -4 \\ -4 & 4 \end{bmatrix}$

$A^*A = 40 \neq 0$ so it is invertible

Now $(A^*A)^{-1} = \frac{1}{40}\begin{bmatrix} 4 & 4 \\ 4 & 14 \end{bmatrix}$ & $x_0 = (A^*A)^{-1}A^*y$

ie $x_0 = \frac{1}{40}\begin{bmatrix} 4 & 4 \\ 4 & 14 \end{bmatrix}\begin{bmatrix} -3 & -2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 9 \\ 6 \\ 2 \\ 1 \end{bmatrix} = \frac{1}{40}\begin{pmatrix} 4 & 4 \\ 4 & 14 \end{pmatrix}\begin{bmatrix} -38 \\ 18 \end{bmatrix}$

$= \frac{1}{40}\begin{bmatrix} -80 \\ 100 \end{bmatrix} = \begin{bmatrix} -2 \\ 2.5 \end{bmatrix}$ Thus $c = -2,\ d = 2.5$

$\therefore\ y = -2t + 2.5$ is the least square line.

Now the error $E = \|Ax_0 - y\|^2 = .25 + .25 + .25 + .25 = 1$

$Ax_0 - y = \begin{bmatrix} -3 & 1 \\ -2 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 2.5 \end{bmatrix} - \begin{bmatrix} 9 \\ 6 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8.5 \\ 6.5 \\ 2.5 \\ 0.5 \end{bmatrix} - \begin{bmatrix} 9 \\ 6 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0.5 \\ 0.5 \\ -0.5 \end{bmatrix}$

(b) $\{(1,2), (3,4), (5,7), (7,9), (9,12)\}$

Sol $A = \begin{bmatrix} 1 & 1 \\ 3 & 1 \\ 5 & 1 \\ 7 & 1 \\ 9 & 1 \end{bmatrix}$ $y = \begin{bmatrix} 2 \\ 4 \\ 7 \\ 9 \\ 12 \end{bmatrix}$ $A^*A = \begin{bmatrix} 1 & 3 & 5 & 7 & 9 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3 & 1 \\ 5 & 1 \\ 7 & 1 \\ 9 & 1 \end{bmatrix} = \begin{bmatrix} 165 & 25 \\ 25 & 5 \end{bmatrix}$

Now $\det(A^*A) = 200 \neq 0$, so A^*A is invertible.

Now $(A^*A)^{-1} = \frac{1}{200}\begin{pmatrix} 5 & -25 \\ -25 & 165 \end{pmatrix}$ & $X_0 = (A^*A)^{-1}A^*y$

ie $X_0 = \begin{bmatrix} c \\ d \end{bmatrix} = \frac{1}{200}\begin{bmatrix} 5 & -25 \\ -25 & 165 \end{bmatrix}\begin{pmatrix} 1 & 3 & 5 & 7 & 9 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}\begin{bmatrix} 2 \\ 4 \\ 7 \\ 9 \\ 12 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 0.55 \end{bmatrix}$

$\Rightarrow y = ct + d = (1.25)t + 0.55$

(c) $\{(-2,4), (-1,3), (0,1), (1,-1)\ (2,-3)\}$

$A = \begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$, $Y = \begin{bmatrix} 4 \\ 3 \\ 1 \\ -1 \\ -3 \end{bmatrix}$ so $A^*A = \begin{bmatrix} -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}$

Now $\det(A^*A) = 50 \neq 0$ $\therefore$ A^*A is invertible.

$x_0 = (A^*A)^{-1}A^*y = \frac{1}{50}\begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}\begin{bmatrix} -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \\ 1 \\ -1 \\ -3 \end{bmatrix}$

$x_0 = \begin{bmatrix} c \\ d \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}\begin{bmatrix} -18 \\ 4 \end{bmatrix} = \frac{1}{50}\begin{bmatrix} -90 \\ 40 \end{bmatrix} = \begin{bmatrix} -1.8 \\ 0.8 \end{bmatrix}$

so the line $-1.8t + 0.8$ is the least square line.

& the error $E = \|Ax_0 - y\|^2 = \left\|\begin{bmatrix} 0.4 \\ -0.4 \\ -0.2 \\ 0.2 \end{bmatrix}\right\| = .16 + .16 + .04 + .04 = 0.40$

Minimal solution to the system of linear Equations

When a system of linear equations $AX = b$ is consistent then there may be no unique solution. In such cases a solution s to $Ax = b$ is called a minimal solution if $\|s\| \leq \|u\|$ for all other solutions u.

Theorem 6.13: Let $A \in M_{m\times n}(F)$ and $b \in F^m$.

Suppose that $Ax = b$ is consistent. Then the fol. statements are true:

(a) There exists exactly one minimal solution s of $Ax = b$, and $s \in R(L_{A^*})$

(b) The vector s is the only solution to $Ax = b$ that lies in $R(L_{A^*})$ ie if u satisfies $(AA^*)u = b$, then $s = A^*u$

Pf not to be done.

Ex. 3 Consider the system: $x + 2y + z = 4$
$x - y + 2z = -11$
$x + 5y = 19$

find the minimal solution to this system.

Sol Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 5 & 0 \end{bmatrix}$ & $b = \begin{bmatrix} 4 \\ -11 \\ 19 \end{bmatrix}$

first we find some sol. u to $AA^*x = b$

Now $AA^* = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 5 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 5 \\ 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 1 & 11 \\ 1 & 6 & -4 \\ 11 & -4 & 26 \end{pmatrix}$

so we consider the system $AA^*x = b$

$$\left.\begin{aligned} 6x + y + 11z &= 4 \\ x + 6y - 4z &= -11 \\ 11x - 4y + 26z &= 19 \end{aligned}\right]$$

One sol for this is $u = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$

Hence $s = A^*u = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 5 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}$ is the minimal sol. to the given system.

Q22 Find the minimal sol. of the fol. system of eqs

(b)
$$\begin{aligned} x + 2y - z &= 1 \\ 2x + 3y + z &= 2 \\ 4x + 7y - z &= 4 \end{aligned}$$

Sol. $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 4 & 7 & -1 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

$AA^* = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 4 & 7 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ -1 & 1 & -1 \end{bmatrix} = \begin{pmatrix} 6 & 7 & 19 \\ 7 & 14 & 28 \\ 19 & 28 & 66 \end{pmatrix}$

$|AA^*| = 0$ $\therefore$ $AA^*x = b$ has many sol

consider $\begin{bmatrix} 6 & 7 & 19 & 1 \\ 7 & 14 & 28 & 2 \\ 19 & 28 & 66 & 4 \end{bmatrix} \sim \begin{bmatrix} 6 & 7 & 19 & 1 \\ 1 & 7 & 9 & 1 \\ 1 & 7 & 9 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 7 & 9 & 1 \\ 0 & 35 & 35 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 7 & 9 & 1 \\ 0 & 7 & 7 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\begin{aligned} x + 7y + 9z &= 1 \\ 7y + 7z &= 1 \end{aligned}$$

Put $z = 0$, $y = 1/7$, $x = 0$

$\therefore$ $u = \begin{bmatrix} 0 \\ 1/7 \\ 0 \end{bmatrix}$ is one sol. of $(AA^*)x = b$

Hence $s = A^*u = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1/7 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/7 \\ 3/7 \\ 1/7 \end{bmatrix}$

(a) $x + 2y - z = 12$

$A = [1, 2, -1]$

$A^* = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$AA^* = [1\ 2\ -1] \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = 6$

$AA^*x = b$

$\Rightarrow 6x = 12 \Rightarrow x = 2$

$\therefore$ $s = A^*u$

$= \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}(2) = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$

(c) $\left.\begin{array}{l} x+y-z=0 \\ 2x-y+z=3 \\ x-y+z=2 \end{array}\right]$ $A=\begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ $b=\begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$

$$AA^* = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 6 & 4 \\ -1 & 4 & 3 \end{bmatrix}$$

Here $|AA^*| = 0$

$\therefore$ Many solutions possible.

Consider the aug. matrix of $(AA^*)(x) = b$ ie $\begin{bmatrix} 3 & 0 & -1 & 0 \\ 0 & 6 & 4 & 3 \\ -1 & 4 & 3 & 2 \end{bmatrix}$

$$\sim \begin{bmatrix} 3 & 0 & -1 & 0 \\ 0 & 6 & 4 & 3 \\ 0 & 12 & 8 & 6 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & -1 & 0 \\ 0 & 6 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} 3x - z = 0 \\ 6y + 4z = 3 \end{array}$$

Take $z=0$, then $x=0$ & $y = 1/2$

$\therefore$ One sol. is $\begin{bmatrix} 0 \\ 1/2 \\ 0 \end{bmatrix}$

$$\therefore s = (A^*)u = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1/2 \\ 1/2 \end{bmatrix}$$

(d) $\begin{array}{l} x+y+z-w=1 \\ 2x-y+w=1 \end{array}$ Let $A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 2 & -1 & 0 & 1 \end{bmatrix}$, $b=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\therefore AA^* = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 2 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}, \quad |AA^*| = 24 \neq 0$$

$$\therefore x = (AA^*)^{-1} b = \frac{1}{24}\begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/6 \end{bmatrix}$$

$$\therefore s = A^*u = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1/4 \\ 1/6 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}+\frac{2}{6} \\ 1/4 - 1/6 \\ 1/4 \\ -1/4 + \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 7/12 \\ 1/12 \\ 1/4 \\ -1/12 \end{bmatrix}$$ Ans.

Q3 sol (a) $V = \mathbb{R}^2$, $T\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a+b \\ a-3b \end{bmatrix}$, $[T]_\beta = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$

$$[T^*]_\beta = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix} \Rightarrow T^*\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a+b \\ a-3b \end{bmatrix} \text{ then } T^*\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 11 \\ -12 \end{bmatrix}$$

(b) $V = \mathbb{C}^2$, $T(z_1, z_2) = (2z_1 + i z_2, (1-i)z_1)$, $\beta = \{e_1, e_2\}$

$$[T]_\beta = \begin{bmatrix} 2 & i \\ 1-i & 0 \end{bmatrix}, \quad [T^*]_\beta = \begin{bmatrix} 2 & 1+i \\ -i & 0 \end{bmatrix} \Rightarrow T^*\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 2z_1 + (1+i)z_2 \\ -i z_1 \end{bmatrix}$$

Then $T^*\begin{pmatrix} 3-i \\ 1+2i \end{pmatrix} = \begin{bmatrix} 2(3-i) + (1+i)(1+2i) \\ -i(3-i) \end{bmatrix} = \begin{bmatrix} 6-2i-1+3i \\ -(1+3i) \end{bmatrix} = \begin{bmatrix} 5+i \\ -(1+3i) \end{bmatrix}$