Section 6.1 : Inner Products and Norms

Def: Let V be a vector space over $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$)

An inner product on V is a function $\langle , \rangle : V \times V \to \mathbb{F}$ that assigns every ordered pair of vectors x, y in V a scalar in F denoted by $\langle x, y \rangle$ such that

(i) $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

(ii) $\langle cx, y \rangle = c\langle x, y \rangle$, $c \in \mathbb{F}$

(iii) $\overline{\langle x, y \rangle} = \langle y, x \rangle$

(iv) $\langle x, x \rangle > 0$ if $x \neq 0$

then $(V \langle , \rangle)$ is called an inner product space. (IPS)

Remarks: (1) (i) and (ii) are equivalent to

$$\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle \quad \alpha, \beta \in \mathbb{F}$$

$\Rightarrow$ inner product is linear in first component.

(2) By (iii), $\overline{\langle x, x \rangle} = \langle x, x \rangle$

$\Rightarrow \langle x, x \rangle =$ real number

$\therefore$ (iv) is meaningful.

(3) $\langle x, \alpha y + \beta z \rangle = \overline{\langle \alpha y + \beta z, x \rangle}$

$= \overline{\alpha \langle y, x \rangle + \beta \langle z, x \rangle}$

$= \bar{\alpha} \overline{\langle y, x \rangle} + \bar{\beta} \overline{\langle z, x \rangle}$

$= \bar{\alpha} \langle x, y \rangle + \bar{\beta} \langle x, z \rangle$

$\Rightarrow$ inner product is conjugate linear in 2nd component.

(4) Dot product in $\mathbb{R}^2$ or $\mathbb{R}^3$ is also an innerproduct

$\because$ if $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ then $x \cdot y = x_1y_1 + x_2y_2 + x_3y_3$

$= \langle x, y \rangle$ in $\mathbb{R}$

Also $\langle x, y \rangle = \langle y, x \rangle$ if the field is real.

Ex 1. For $x=(a_1, a_2, \ldots a_n)$ and $y=(b_1, b_2, \ldots$

define $\langle x, y\rangle = \sum_{i=1}^{n} a_i \overline{b_i}$ — (1)

It is easy to verify that $\langle , \rangle$ satisfies conditions (i) $\Rightarrow$ (IV).

As a particular case, for $x=(1+i, 4)$ & $y=(2-3i, 4+5i)$ in $\mathbb{C}^2$, $\langle x, y\rangle = (1+i)(2+3i)+4(4-5i) = 15-15i$

Inner product (1) is called the standard inner product on F^n.

Ex 2. If $\langle x, y\rangle$ is any innerproduct on a v.s. V & $r>0$, define $\langle x, y\rangle' = r\langle x, y\rangle$

$\langle , \rangle'$ is also an inner product on V as

(i) $\langle x+y, z\rangle' = r\langle x+y, z\rangle = r[\langle x, z\rangle + \langle y, z\rangle]$
$= r\langle x, z\rangle + r\langle y, z\rangle = \langle x, z\rangle' + \langle y, z\rangle'$

(ii) $\langle cx, y\rangle' = r\langle cx, y\rangle = rc\langle x, y\rangle = c\, r\langle x, y\rangle$
$= c\langle x, y\rangle'$

(iii) $\overline{\langle x, y\rangle'} = \overline{r\langle x, y\rangle} = r\overline{\langle x, y\rangle} = r\langle y, x\rangle = \langle y, x\rangle'$

(IV) $\langle x, x\rangle' = r\langle x, x\rangle > 0$ as $r>0$ & $x \neq 0$.

Ex. 3 Let $V = C([0,1])$, the vector space of real valued continuous functions on $[0,1]$. For $f, g \in V$, define $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$

Verify that $\langle , \rangle$ is an Inner product on V.

Def Let $A \in M_{m\times n}(F)$. We define the conjugate transpose or adjoint of A to be the $n\times m$ matrix A^* such that $(A^*)_{ij} = \overline{A_{ji}}$ $\forall i \& j$.

eg If $A = \begin{pmatrix} i & 1+5i \\ 2 & 4+3i \end{pmatrix}$ then $A^* = \begin{pmatrix} -i & 2 \\ 1-5i & 4-3i \end{pmatrix}$

Remark: If $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ are in F^n, then

$$\langle x, y\rangle = y^* x = \sum_{i=1}^{n} x_i \overline{y_i} \quad \left[\because y^*x = [\overline{y_1} \cdots \overline{y_n}]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^{n} x_i \overline{y_i}\right]$$

Ex. 5. Let $V = M_{n\times n}(F)$ & define $\langle A, B\rangle = tr(B^*A)$ for $A, B \in V$. Then

(i) $\langle A+B, C\rangle = tr(C^*(A+B)) = tr(C^*A + C^*B)$
$= tr(C^*A) + tr(C^*B) = \langle A, C\rangle + \langle B, C\rangle$

(ii) $\langle cA, B\rangle = tr(B^*(cA)) = tr(cB^*A) = c\, tr(B^*A)$
$= c\langle A, B\rangle$

(iii) $\overline{\langle A, B\rangle} = \overline{tr(B^*A)} = tr(\overline{B^*A}) = tr(\overline{B^*}\,\overline{A}) = tr(A^*B)$
$= \langle B, A\rangle$

(IV) $\langle A, A\rangle = tr(A^*A) = \sum_{i=1}^{n} (A^*A)_{ii} = \sum_{i=1}^{n}\sum_{k=1}^{n} (A^*)_{ik} A_{ki}$
$= \sum_{i=1}^{n}\sum_{k=1}^{n} \overline{A_{ki}} A_{ki} = \sum_{i=1}^{n}\sum_{k=1}^{n} |A_{ki}|^2$

Now if $A \neq 0$, then $A_{ki} \neq 0$ for some k & i & so $\langle A, A\rangle > 0$.

Inner product in this example is called the Frobenius inner product.

Theorem 6.1 Let V be an IPS. Then for $x, y, z \in V$ and $c \in F$, the fol. statements are true

(i) $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$

(ii) $\langle x, cy \rangle = \bar{c} \langle x, y \rangle$

(iii) $\langle x, 0 \rangle = \langle 0, x \rangle = 0$ (iv) $\langle x, x \rangle = 0$ iff $x = 0$

(v) If $\langle x, y \rangle = \langle x, z \rangle \ \forall x \in V$ then $y = z$.

Pf: We have

(i) $\langle x, y+z \rangle = \overline{\langle y+z, x \rangle} = \overline{\langle y, x \rangle + \langle z, x \rangle}$

$= \overline{\langle y, x \rangle} + \overline{\langle z, x \rangle} = \langle x, y \rangle + \langle x, z \rangle$

(ii) We have $\langle x, cy \rangle = \overline{\langle cy, x \rangle} = \bar{c} \overline{\langle y, x \rangle} = \bar{c} \langle x, y \rangle$

(iii) $\langle x, 0 \rangle = \langle x, 0.0 \rangle = \bar{0} \langle x, 0 \rangle = 0 \langle x, 0 \rangle = 0$

Similarly $\langle 0, x \rangle = \overline{\langle x, 0 \rangle} = \bar{0} = 0$

(iv) We have $\langle x, x \rangle = 0 \Leftrightarrow x = 0$

suppose $\langle x, x \rangle = 0$ then $x = 0$ [for if $x \neq 0$ then $\langle x, x \rangle > 0$ *]

Conversely, let $x = 0$ then $\langle x, x \rangle = 0$ [by (iii)]

(v) ~~If~~ let $\langle x, y \rangle = \langle x, z \rangle \ \forall x \in V$

then $\langle x, y-z \rangle = \langle x, y \rangle - \langle x, z \rangle = 0 \ \forall x \in V$

In particular when $x = y - z$

$0 = \langle y-z, y-z \rangle$

$\therefore$ by (iii), we get $y - z = 0 \Rightarrow y = z$

Remark: An inner product space V over $\mathbb{C}$ is called complex inner product space whereas over $\mathbb{R}$ it is called real inner product space.

Remark (2) If V has an innerproduct $\langle x, y\rangle$ and W is a subspace of V, then W is also an IPS when the same function $\langle x, y\rangle$ is restricted to the vectors $x, y \in W$.

(3) Two distinct inner products on a given vector space produce two distinct inner product spaces.

eg. If $V = P(R)$ then both

$$\langle f(x), g(x)\rangle_1 = \int_0^1 f(t)\, g(t)\, dt \quad \& \quad \langle f(x), g(x)\rangle_2 = \int_{-1}^{1} f(t)\, g(t)\, dt$$

are innerproducts on V but they produce two different inner product spaces.

$\therefore$ if $f(x) = x$, $g(x) = x^2$ then $\langle f, g\rangle_1 = \int_0^1 t^3\, dt = \frac{1}{4}$

& $\langle f, g\rangle_2 = \int_{-1}^{1} t^3\, dt = 0$

<u>Def:</u> <u>Norm of a vector</u>: Let V be an IPS, For $x \in V$, We define the norm or length of x by

$$\|x\| = \sqrt{\langle x, x\rangle}$$

Ex 6. Let $V = \mathbb{F}^n$. If $x = (a_1, a_2, \ldots, a_n)$, then

$$\|x\| = \|(a_1, a_2, \ldots a_n)\| = \left[\sum_{i=1}^{n} |a_i|^2\right]^{1/2}$$

is the Euclidean def. of length.

Note that if $n = 1$ then $\|x\| = |a_1| = |x|$ & if $n = 2$ then $\|x\| = \left[|a_1|^2 + |a_2|^2\right]^{1/2}$ = length or magnitude of the vector x

<u>Theorem 6.2</u>: Let V be an IPS over F. Then $\forall\, x, y \in V$ and $c \in F$, the fol. statements are true:

(a) $\|cx\| = |c|\, \|x\|$ (b) $\|x\| = 0$ iff $x = 0$. In any case $\|x\| \geq 0$

(c) (Cauchy-Schwarz - Inequality) $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$.

(d) (Triangle inequality) $\|x + y\| \leq \|x\| + \|y\|$.

Proof: (a) we have

$$\|cx\|^2 = \langle cx, cx\rangle = c\langle x, cx\rangle = c.\bar{c}\langle x, x\rangle$$
$$= |c|^2\|x\|^2$$

$$\Rightarrow \|cx\|^2 = |c|\|x\|$$

(b) Let $\|x\| = 0 \Rightarrow \sqrt{\langle x, x\rangle} = 0 \Rightarrow \langle x, x\rangle = 0 \Rightarrow x = 0$

Conversely, $x = 0 \Rightarrow \langle x, x\rangle = 0$ ie $\|x\|^2 = 0 \Rightarrow \|x\| = 0$

As $\langle x, x\rangle > 0$ if $x \neq 0$ so $\|x\| \geq 0$.

(c) If $y = 0$, then the result is immediate.

$\therefore$ let $y \neq 0$ then $\|y\| \neq 0$. Also for any $c \in F$

$$0 \leq \|x - cy\|^2 = \langle x - cy, x - cy\rangle = \langle x, x - cy\rangle - \langle cy, x - cy\rangle$$
$$= \langle x, x\rangle - \bar{c}\langle x, y\rangle - c\langle y, x\rangle + c\bar{c}\langle y, y\rangle$$

Let's take $c = \dfrac{\langle x, y\rangle}{\langle y, y\rangle}$ then

$$0 \leq \|x\|^2 - \overline{\frac{\langle x, y\rangle}{\langle y, y\rangle}}\langle x, y\rangle - \frac{\langle x, y\rangle}{\langle y, y\rangle}\langle y, x\rangle + \frac{\langle x, y\rangle}{\langle y, y\rangle}\overline{\frac{\langle x, y\rangle}{\langle y, y\rangle}}\langle y, y\rangle$$

$$= \|x\|^2 - \frac{|\langle x, y\rangle|^2}{\|y\|^2} - \cancel{\frac{|\langle x, y\rangle|^2}{\|y\|^2}} + \cancel{\frac{|\langle x, y\rangle|^2}{\|y\|^2}}$$

$$\Rightarrow 0 \leq \|x\|^2\|y\|^2 - |\langle x, y\rangle|^2 \Rightarrow |\langle x, y\rangle|^2 \leq \|x\|^2\|y\|^2$$

$$\Rightarrow |\langle x, y\rangle| \leq \|x\| \, \|y\|.$$

(d) we have $\|x + y\|^2 = \langle x + y, x + y\rangle = \langle x, x + y\rangle + \langle y, x + y\rangle$

$$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle$$
$$= \|x\|^2 + \{\langle x, y\rangle + \overline{\langle x, y\rangle}\} + \|y\|^2$$
$$= \|x\|^2 + 2\,\mathrm{Re}\,\langle x, y\rangle + \|y\|^2$$
$$\leq \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2$$
$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$$

Hence, $\|x + y\| \leq \|x\| + \|y\|$

Ex 7. Let $V = \mathbb{F}^n$; $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

$x = (a_1, a_2, \ldots a_n)$, $y = (b_1, b_2, \ldots, b_n) \in V$

then $|\langle x, y\rangle| = \left|\sum_{i=1}^{n} a_i \bar{b}_i\right| \leq \|x\| \, \|y\|$ (by Cauchy Schwarz inequality)

$$= \left(\sum |a_i|^2\right)^{1/2} \left(\sum |b_i|^2\right)^{1/2}$$

ie
$$\left|\sum_{i=1}^{n} a_i \bar{b}_i\right| \leq \left(\sum |a_i|^2\right)^{1/2} \left(\sum |b_i|^2\right)^{1/2}$$

&
$$\|x+y\| = \left[\langle x+y, x+y\rangle\right]^{1/2} = \left(\sum_{i=1}^{n} |a_i + b_i|^2\right)^{1/2} \leq \|x\| + \|y\|$$
$$= \left(\sum |a_i|^2\right)^{1/2} + \left(\sum |b_i|^2\right)^{1/2}$$

ie
$$\left(\sum_{i=1}^{n} |a_i + b_i|^2\right)^{1/2} \leq \left(\sum_{i=1}^{n} |a_i|^2\right)^{1/2} + \left(\sum_{i=1}^{n} |b_i|^2\right)^{1/2}.$$

Definitions ① vectors $x, y \in V$ are said to be orthogonal or perpendicular if $\langle x, y\rangle = 0$

Observe that $\langle x, y\rangle = 0 \Leftrightarrow \langle y, x\rangle = 0$

Def ② A subset S of V is called orthogonal iff $\langle x, y\rangle = 0 \;\forall\; x, y \in S$.

Def ③ A vector x in V is a unit vector if $\|x\| = 1$.

Def ④ A subset S of V is called orthonormal if S is orthogonal and consists entirely of unit vectors.

Remark; If $S = \{v_1, v_2 \ldots\}$, then S is orthonormal iff $\langle v_i, v_j\rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$

Def; The process of multiplying a non-zero vector by the reciprocal of its length is called normalizing.

Ex 8. In F^3, $S = \{(1,1,0), (1,-1,1), (-1,1,2)\}$ is an orthogonal set of non-zero vectors,

whereas $S_1 = \left\{ \frac{1}{\sqrt{2}}(1,1,0), \frac{1}{\sqrt{3}}(1,-1,1), \frac{1}{\sqrt{6}}(-1,1,2) \right\}$

is an orthonormal set.

Ex. 9. Let $H = \mathbb{C}[0,1]$ = the vector space of cts complex valued fns defined on $[0, 2\pi]$

define $\langle f, g \rangle$ on H by $\langle f, g \rangle = \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}\,dt$

For any integer n, let $f_n(t) = e^{int}$, $0 \le t \le 2\pi$

Let $S = \{f_n : n \text{ is an integer}\}$. Clearly $S \subseteq H$.

We shall show that S is orthonormal.

For $m \neq n$, we have

$$\langle f_m, f_n \rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{imt}\,\overline{e^{int}}\,dt = \frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)t}\,dt$$

$$= \frac{1}{2\pi(m-n)} e^{i(m-n)t}\Big|_0^{2\pi} = 0$$

Also $\langle f_n, f_n \rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{i(n-n)t}\,dt = \frac{1}{2\pi}\int_0^{2\pi} 1.\,dt = 1$

Exercises

Q.2. $x = (2, 1+i, i)$ & $y = (2-i, 2, 1+2i)$ be the vectors in $\mathbb{C}^3$. Compute $\langle x, y \rangle$, $\|x\|$, $\|y\|$ and $\|x+y\|$. Then verify both the cauchy-schwarz inequality and triangle inequality.

Sol $\langle x, y \rangle = 2.(2+i) + (1+i)2 + i(1-2i) = 4+2+4i+i+2 = 8+5i$

$\|x\| = \sqrt{2\times 2 + (1+i)(1-i) + i(-i)} = \sqrt{4+2+1} = \sqrt{7}$

$\|y\| = \sqrt{(2-i)(2+i) + 2.2 + (1+2i)(1-2i)} = \sqrt{14}$

$x+y = (4-i, 3+i, 1+3i)$ & $\|x+y\| = \sqrt{(4-i)(4+i) + (3+i)(3-i) + (1+3i)(1-3i)}$

$= \sqrt{17+10+10} = \sqrt{37}$

$|\langle x, y \rangle| = \sqrt{8^2+5^2} = \sqrt{64+25} = \sqrt{89} < \sqrt{98} = \sqrt{7}\times\sqrt{14} = \|x\|\,\|y\|$

$\|x+y\| = \sqrt{37}$ & $\|x\| + \|y\| = \sqrt{7} + \sqrt{14}$

Q3. In $C[0,1]$, let $f(t) = t$ & $g(t) = e^t$. Compute $\langle f, g\rangle = \int_0^1 f(t)\,g(t)\,dt$, $\|f\|$, $\|g\|$ and $\|f+g\|$. Then verify both the Cauchy Schwarz inequality & the triangle inequality

Sol $\langle f, g\rangle = \int_0^1 t e^t\,dt = (t-1)e^t\Big|_0^1 = 1$ & $|\langle f, g\rangle| = 1$

$\|f\| = \sqrt{\langle f, f\rangle} = \left(\int_0^1 t^2\,dt\right)^{1/2} = \left(\frac{t^3}{3}\Big|_0^1\right)^{1/2} = \frac{1}{\sqrt{3}}$

$\|g\| = \sqrt{\langle g, g\rangle} = \left(\int_0^1 e^{2t}\,dt\right)^{1/2} = \sqrt{\frac{e^{2t}}{2}\Big|_0^1} = \sqrt{\left(\frac{e^2-1}{2}\right)}$

$\|f\|\,\|g\| = \sqrt{\frac{e^2-1}{6}}$ Thus $|\langle f, g\rangle| < \sqrt{\frac{e^2-1}{6}}$

Q4 (b) $A = \begin{pmatrix} 1 & 2+i \\ 3 & i \end{pmatrix}$ & $B = \begin{pmatrix} 1+i & 0 \\ i & -i \end{pmatrix}$ $A^* = \begin{pmatrix} 1 & 3 \\ 2-i & -i \end{pmatrix}$ & $B^* = \begin{pmatrix} 1-i & -i \\ 0 & i \end{pmatrix}$

Sol

$\|A\| = \sqrt{\langle A, A\rangle} = \sqrt{tr(A^*A)} = \sqrt{tr\begin{pmatrix} 10 & 2+4i \\ 2-4i & 6 \end{pmatrix}} = \sqrt{16} = 4$

$\|B\| = \sqrt{\langle B, B\rangle} = \sqrt{tr(B^*B)} = \sqrt{tr\begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}} = \sqrt{4} = 2$

$\langle A, B\rangle = tr(B^*A) = tr\begin{pmatrix} 1-4i & 3-i \\ 3i & -1 \end{pmatrix} = -4i$

Q5 In $\mathbb{C}^2$, show that $\langle x, y\rangle = xAy^*$ is an inner product, where $A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}$ compute $\langle x, y\rangle$ for $x = (1-i,\ 2+3i)$ & $y = (2+i,\ 3-2i)$

Sol (i) $\langle x+z, y\rangle = (x+z)Ay^* = xAy^* + zAy^* = \langle x, y\rangle + \langle z, y\rangle$

(ii) $\langle cx, y\rangle = (cx)Ay^* = c(xAy^*) = c\langle x, y\rangle$

(iii) $\overline{\langle x, y\rangle} = \overline{xAy^*} = \bar{x}\bar{A}\bar{y^*} = \bar{x}\bar{A}y^t = (yA^*\bar{x}^t)^t = yAx^*$
$= \langle y, x\rangle$

(iv) $\langle x, x\rangle = xAx^* > 0$ if $x \neq 0$

$\langle x, y\rangle = (1-i,\ 2+3i)\begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}\begin{bmatrix} 2-i \\ 3+2i \end{bmatrix} = (1-i,\ 2+3i)\begin{bmatrix} 2-i+3i-2 \\ -2i-1+6+4i \end{bmatrix}$

$= (1-i,\ 2+3i)\begin{bmatrix} 2i \\ 5+2i \end{bmatrix} = 2i+2+10+4i+15i-6 = 6+21i$

Q9 Let β be a basis for a FDIPS.

(a) Prove that if $\langle x, z\rangle = 0 \ \forall z \in \beta$ then $x = 0$

(b) Prove that if $\langle x, z\rangle = \langle y, z\rangle \ \forall z \in \beta$ then $x = y$

Sol Let $\beta = \{v_1, v_2, \dots, v_n\}$

By assumption $\langle x, v_i\rangle = 0 \quad \forall i = 1, 2, \dots, n$

Let $v \in V \Rightarrow v = \sum_{i=1}^{n} \alpha_i v_i$ then $\langle x, v\rangle = \langle x, \sum \alpha_i v_i\rangle$

$= \sum \bar{\alpha}_i \langle x, v_i\rangle = 0$

$\Rightarrow \langle x, v\rangle = 0 \quad \forall v \in V.$

In particular $\langle x, x\rangle = 0 \Rightarrow x = 0.$

(b) Given $\langle x, z\rangle = \langle y, z\rangle \quad \forall z \in \beta$

$\Rightarrow \langle x - y, z\rangle = 0 \quad \forall z \in \beta$

By part (a) $x - y = 0 \Rightarrow x = y.$

Q 10 Let V be an IPS & suppose that x & y are orthogonal vectors in V. Prove that $\|x+y\|^2 = \|x\|^2 + \|y\|^2$. Deduce the pythagorean Theorem in $\mathbb{R}^2$.

Sol $LHS = \|x+y\|^2 = \langle x+y, x+y\rangle = \langle x, x+y\rangle + \langle y, x+y\rangle$

$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle$

$= \|x\|^2 + 0 + 0 + \|y\|^2 = \|x\|^2 + \|y\|^2 = RHS.$

If x & y ~~are vectors in~~ represents sides of a Δ $\mathbb{R}^2$, then $x+y$ represents the hypotenuse as x & y are $\perp$ & we have

$\|x+y\|^2 = \|x\|^2 + \|y\|^2$

Q11 Prove the parallelogram Law on an IPS V ie show that

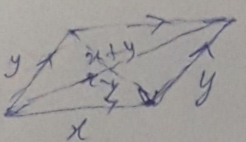
$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \forall x, y \in V.$

what does this eq. state about parallelograms in $\mathbb{R}^2$.

Ans 11. $LHS = \|x+y\|^2 + \|x-y\|^2$

$= \langle x+y, x+y \rangle + \langle x-y, x-y \rangle$

$= \langle x, x+y \rangle + \langle y, x+y \rangle + \langle x, x-y \rangle - \langle y, x-y \rangle$



$= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle + \langle x, x \rangle$

$- \langle x, y \rangle - \langle y, x \rangle + \langle y, y \rangle$

$= 2(\|x\|^2 + \|y\|^2) = RHS$

$\Rightarrow$ sum of squares of the diagonals of a llgm

= sum of squares of its four sides.

Q12 Let $\{v_1, v_2, \ldots, v_k\}$ be an orthogonal set in V and let $a_1, a_2, \ldots a_k$ be scalars. prove that

$$\left\| \sum_{i=1}^{k} a_i v_i \right\|^2 = \sum_{i=1}^{k} |a_i|^2 \|v_i\|^2.$$

Sol $\left\| \sum_{i=1}^{k} a_i v_i \right\|^2 = \left\langle \sum_{i=1}^{k} a_i v_i, \sum_{j=1}^{k} a_j v_j \right\rangle$

$= \sum_{i=1}^{k} a_i \left\langle v_i, \sum_{j=1}^{k} a_j v_j \right\rangle$

$= \sum_{i=1}^{k} a_i \left\{ \sum_{j=1}^{k} \bar{a}_j \langle v_i, v_j \rangle \right\}$

$= \sum_{i=1}^{k} a_i \bar{a}_i \langle v_i, v_i \rangle$ $\quad [\langle v_i, v_j \rangle = 0$ if $i \neq j]$

$= \sum_{i=1}^{k} |a_i|^2 \|v_i\|^2.$

Q. 15 Prove that if V is an IPS, then $|\langle x, y \rangle| = \|x\| \cdot \|y\|$ iff one of the vectors x or y is a multiple of the other.

OR show that cauchy schwarz inequality is equality iff x or y is a multiple of the other

Sol suppose $|\langle x, y \rangle| = \|x\| \, \|y\|$

If $x = 0$ then $x = 0 . y \Rightarrow x$ is a multiple of y.

If $x \neq 0$, let $z = y - \frac{\langle y, x\rangle}{\|x\|^2} \cdot x$

Consider $\langle z, z\rangle = \left\langle y - \frac{\langle y, x\rangle}{\|x\|^2} \cdot x ,\ y - \frac{\langle y, x\rangle}{\|x\|^2} \cdot x \right\rangle$

$= \left\langle y, y - \frac{\langle y, x\rangle}{\|x\|^2} \cdot x \right\rangle - \frac{\langle y, x\rangle}{\|x\|^2} \left\langle x, y - \frac{\langle y, x\rangle}{\|x\|^2} \cdot x \right\rangle$

$= \langle y, y\rangle - \frac{\overline{\langle y, x\rangle}}{\|x\|^2} \langle y, x\rangle - \frac{\langle y, x\rangle}{\|x\|^2} \langle x, y\rangle + \frac{\langle y, x\rangle}{\|x\|^2} \cdot \frac{\overline{\langle y, x\rangle}}{\|x\|^2} \langle x, x\rangle$

$= \|y\|^2 - \frac{|\langle y, x\rangle|^2}{\|x\|^2} - \frac{|\langle x, y\rangle|^2}{\|x\|^2} + \frac{|\langle y, x\rangle|^2 \cdot \cancel{\|x\|^2}}{\|x\|^2 \cancel{\|x\|^2}}$

$= \|y\|^2 - \frac{|\langle x, y\rangle|^2}{\|x\|^2} = \|y\|^2 - \frac{\|x\|^2 \|y\|^2}{\|x\|^2} = 0$

$\Rightarrow z = 0 \Rightarrow y = \frac{\langle y, x\rangle}{\|x\|^2} x = \alpha x$ where $\alpha = \frac{\langle y, x\rangle}{\|x\|^2}$

ie y is a multiple of x.

Conversely, Let $y = \alpha x$,

$\therefore |\langle x, y\rangle| = |\langle x, \alpha x\rangle| = |\bar{\alpha} \langle x, x\rangle| = |\bar{\alpha}| \|x\|^2 = |\alpha| \|x\|^2$

Now $\|x\| \|y\| = \|x\| \|\alpha x\| = |\alpha| \|x\|^2 = |\langle x, y\rangle|$

(b) Derive a similar result for the equality $\|x + y\| = \|x\| + \|y\|$ and generalize it to the case of n vectors.

Sol Consider $\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + \|y\|^2 + \langle x, y\rangle + \langle y, x\rangle$

Suppose $\|x+y\| = \|x\| + \|y\|$ (ie Δ inequality is equality)

$\Rightarrow \|x+y\|^2 = [\|x\| + \|y\|]^2$

$\Rightarrow \|x\|^2 + \|y\|^2 + \langle x, y\rangle + \langle y, x\rangle = \|x\|^2 + \|y\|^2 + 2\|x\|\|y\|$

$\Rightarrow \langle x, y\rangle + \overline{\langle x, y\rangle} = 2\|x\| \|y\|$

$\Rightarrow 2 \operatorname{Re}\langle x, y\rangle = 2\|x\| \|y\| \Rightarrow \operatorname{Re}\langle x, y\rangle = \|x\| \|y\|$

Let $x \neq 0$ & let $z = y - \frac{\|y\|}{\|x\|} \cdot x$.

Then
$$\langle z, z\rangle = \langle y - \frac{\|y\|}{\|x\|} x, \; y - \frac{\|y\|}{\|x\|} x \rangle$$
$$= \langle y, y\rangle - \frac{\|y\|}{\|x\|}\langle y, x\rangle - \frac{\|y\|}{\|x\|}\langle x, y\rangle + \frac{\|y\|^2}{\|x\|^2}\langle x, x\rangle$$
$$= \langle y, y\rangle - 2\frac{\|y\|}{\|x\|}\,\text{real}\,\langle x, y\rangle + \|y\|^2$$
$$= \|y\|^2 - 2\frac{\|y\|}{\cancel{\|x\|}}\cancel{\|x\|}\,\|y\| + \|y\|^2 = 2\|y\|^2 - 2\|y\|^2 = 0$$

$\Rightarrow z = 0 \Rightarrow y = \frac{\|y\|}{\|x\|} x = cx$ where $c = \frac{\|y\|}{\|x\|} \geq 0$

ie y is non-negative multiple of x

If $x = 0$ then $x = 0 \cdot y \Rightarrow x$ is a non-negative multiple of y.

Conversely, let $y = cx$, $c \geq 0$, $c \in \mathbb{R}$

$LHS = \|x + y\| = \|x + cx\| = |1 + c|\,\|x\| = (1 + c)\|x\|$

$RHS = \|x\| + \|y\| = \|x\| + \|cx\| = \|x\| + |c|\,\|x\|$
$= (1 + |c|)\|x\| = (1 + c)\|x\|$

$\Rightarrow LHS = RHS$.

Q17 Let T be a LO on an IPS V & suppose that $\|Tx\| = \|x\| \; \forall x$. Prove that T is one-to-one

Sol T is one-one iff $Tx = 0$ has only the trivial sol $x = 0$.

$T(x) = 0 \Leftrightarrow \|Tx\| = 0 \Leftrightarrow \|x\| = 0 \Leftrightarrow x = 0$

Q19 Let V be an IPS. Prove that

(a) $\|x \pm y\|^2 = \|x\|^2 \pm 2\,\mathrm{Re}\,\langle x, y\rangle + \|y\|^2 \quad \forall x, y \in V$

(b) $\big|\,\|x\| - \|y\|\,\big| \leq \|x - y\| \quad \forall x, y \in V$

Pf (a) $\|x \pm y\|^2 = \|x\|^2 + \|y\|^2 \pm \langle x, y\rangle \pm \langle y, x\rangle$
$= \|x\|^2 + \|y\|^2 \pm 2\,\mathrm{Re}\,\langle x, y\rangle$

(b) we have

$$\|x\| = \|x-y+y\| \le \|x-y\| + \|y\|$$

$\Rightarrow$ $\|x\| - \|y\| \le \|x-y\| \quad \forall\, x, y \in V$ —— (1)

interchanging x & y, we get

$$\|y\| - \|x\| \le \|y-x\| = \|-(x-y)\| = |-1|\,\|x-y\| = \|x-y\|$$

$\forall\, x, y \in V$

Thus $\Rightarrow$ $-[\|x\| - \|y\|] \le \|x-y\| \quad \forall\, x, y \in V$ —— (2)

from (1) & (2), we get $\big|\|x\| - \|y\|\big| \le \|x-y\|$

Q.20 Let V be an IPS over F. Prove the polar identities : For all $x, y \in V$

(a) $\langle x, y\rangle = \frac{1}{4}\|x+y\|^2 - \frac{1}{4}\|x-y\|^2$ if $F = \mathbb{R}$

(b) $\langle x, y\rangle = \frac{1}{4}\sum_{k=1}^{4} i^k \|x+i^k y\|^2$ if $F = \mathbb{C}$ where $i^2 = -1$.

Pf. (a) RHS $= \frac{1}{4}\|x+y\|^2 - \frac{1}{4}\|x-y\|^2$

$= \frac{1}{4}\langle x+y, x+y\rangle - \frac{1}{4}\langle x-y, x-y\rangle$

$= \frac{1}{4}[\|x\|^2 + \|y\|^2 + \langle x, y\rangle + \langle y, x\rangle] - \frac{1}{4}[\|x\|^2 + \|y\|^2 - \langle x, y\rangle - \langle y, x\rangle]$

$= \frac{1}{4} \times 2\langle x, y\rangle + \frac{1}{4} \times 2\langle x, y\rangle = \langle x, y\rangle \qquad [\because \langle x, y\rangle = \langle y, x\rangle]$

(b) RHS $= \frac{1}{4}[i\|x+iy\|^2 - \|x-y\|^2 - i\|x-iy\|^2 + \|x+y\|^2]$

$= \frac{1}{4} i [\|x+iy\|^2 - \|x-iy\|^2] + \frac{1}{4}[\|x+y\|^2 - \|x-y\|^2]$

$= \frac{1}{4} i [\langle x+iy, x+iy\rangle - \langle x-iy, x-iy\rangle] + \frac{1}{4}[\langle x+y, x+y\rangle - \langle x-y, x-y\rangle]$

$= \frac{1}{4} i [\cancel{\langle x, x\rangle} + i\langle y, x\rangle - i\langle x, y\rangle + \cancel{\langle y, y\rangle} - (\cancel{\langle x, x\rangle} - i\langle y, x\rangle + i\langle x, y\rangle + \cancel{\langle y, y\rangle})]$
$+ \frac{1}{4}[\cancel{\langle x, x\rangle} + \langle x, y\rangle + \langle y, x\rangle + \cancel{\langle y, y\rangle} - (\cancel{\langle x, x\rangle} - \langle x, y\rangle - \langle y, x\rangle + \cancel{\langle y, y\rangle})]$

$= \frac{1}{4} i [2i\langle y, x\rangle - 2i\langle x, y\rangle] + \frac{1}{4}[2\langle x, y\rangle + 2\langle y, x\rangle]$

$= -\frac{1}{2}\langle y, x\rangle + \frac{1}{2}\langle x, y\rangle + \frac{1}{2}\langle x, y\rangle + \frac{1}{2}\langle y, x\rangle = \langle x, y\rangle$