Reduction formulae

We first recall the integration by parts formula for definite integrals

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

where u & v are functions of x & the limits of integration are limits on the variable x.

Integration by parts can be used to derive reduction formulas for integrals. Reduction formulas express an integral involving a power of function in terms of an integral that involves a lower power of that function. For example, if n is a positive integer and $n \geq 2$, then integration by parts can be used to obtain the reduction formulas.

example: Show that

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx.$$

Solution: Let $I_n = \int \sin^n x\,dx$

Then $I_n = \int \sin^n x\,dx = \int \sin^{n-1}x\,\sin x\,dx$

let $u = \sin^{n-1}x \Rightarrow du = (n-1)\sin^{(n-2)}x\cos x\,dx$

let $v = \cos x \Rightarrow dv = -\sin x\,dx$

$\Rightarrow I_n = -\int u\,dv$ (Using by parts formula)

$= -\left[uv - \int v\,du\right]$

$= -\left[\sin^{n-1}x\cos x - \int \cos x\,(n-1)\sin^{(n-2)}x\cos x\,dx\right]$

$$= -\left[\sin^{n-1}x\cos x - (n-1)\int \cos^2 x \sin^{n-2}x\, dx\right]$$

$$= -\left[\sin^{n-1}x\cos x - (n-1)\int (1-\sin^2 x)\sin^{n-2}x\, dx\right]$$

$$= -\left[\sin^{n-1}x\cos x - (n-1)\int \sin^{n-2}x\, dx + (n-1)\int \sin^n x\, dx\right]$$

$$I_n = -\sin^{n-1}x\cos x + (n-1)I_{n-2} - (n-1)I_n \quad \text{where } I_{n-2} = \int \sin^{n-2}x\, dx$$

$$\Rightarrow \{1+(n-1)\}I_n = -\sin^{n-1}x\cos x + (n-1)I_{n-2}$$

$$\Rightarrow nI_n = -\sin^{n-1}x\cos x + (n-1)I_{n-2}$$

$$\Rightarrow \int \sin^n x\, dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{(n-1)}{n}\int \sin^{n-2}x\, dx.$$

Hence, proved.

Q Show that

$$\int \cos^n x\, dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{(n-1)}{n}\int \cos^{n-2}x\, dx.$$

Solution: Let $I_n = \int \cos^n x\, dx$.

$$= \int \cos^{n-1}x\cos x\, dx$$

Let $u = \cos^{n-1}x \Rightarrow du = -(n-1)\cos^{n-2}x\sin x\, dx$

let $v = \sin x \Rightarrow dv = \cos x\, dx$.

Then $I_n = \int u\,dv$

$$= uv - \int v\,du$$

$$= \cos^{n-1}x\sin x + \int \sin x\,(n-1)\cos^{n-2}x\sin x\, dx$$

$$= \cos^{n-1}x\sin x + (n-1)\int \sin^2 x\cos^{n-2}x\, dx$$

$$= \cos^{n-1}x\sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2}x\, dx$$

$$I_n = \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x\,dx - (n-1)\int \cos^n x\,dx$$

$$= \cos^{n-1}x \sin x + (n-1)I_{n-2} - (n-1)I_n$$ where $I_{n-2} = \int \cos^{n-2}x\,dx$

$$\Rightarrow [1+(n-1)]I_n = \cos^{n-1}x \sin x + (n-1)I_{n-2}$$

$$\Rightarrow I_n = \frac{1}{n}\cos^{n-1}x \sin x + \frac{(n-1)}{n} I_{n-2}$$

Hence, proved.

Q. Evaluate $\int \cos^4 x\,dx$.

Taking $n=4$ in the reduction formula for $\int \cos^n x\,dx$ we have

$$\int \cos^4 x\,dx = \frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\int \cos^2 x\,dx$$

$$= \frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\left(\frac{1}{2}\cos x \sin x + \frac{1}{2}\int dx\right)$$

use reduction formula taking $n=2$

$$= \frac{1}{4}\cos^3 x \sin x + \frac{3}{8}\cos x \sin x + \frac{3}{8}x + c.$$

Q. ~~Evaluate~~ ~~[illegible]~~

If m and n are positive integers, then the integral

$$\int \sin^m x \cos^n x\,dx$$

can be evaluated by one of the three procedures stated in table given below, depending on whether m and n are odd or even.

$\int \sin^m x \cos^n x \, dx$	Procedure	Relevant Identities
n odd	• Split off a factor of $\cos x$ • Apply the relevant identity • Make the substitution $u = \sin x$	$\cos^2 x = 1 - \sin^2 x$
m odd	• Split off a factor of $\sin x$ • Apply the relevant identity • Make the substitution $u = \cos x$	$\sin^2 x = 1 - \cos^2 x$
m even n even	• Use the relevant identities to reduce the powers on $\sin x$ and $\cos x$	$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

Q Evaluate $\int \sin^4 x \cos^5 x \, dx$.

Since n is odd, therefore we split off a factor of $\cos x$,

$$\int \sin^4 x \cos^5 x \, dx = \int \sin^4 x \cos^4 x \cos x \, dx$$

$$= \int \sin^4 x \, (1 - \sin^2 x)^2 \cos x \, dx$$

let $u = \sin x$ then $du = \cos x \, dx$

$$\int \sin^4 x \cos^5 x \, dx = \int u^4 (1 - u^2)^2 \, du$$

$$= \int (u^4 - 2u^6 + u^8) \, du$$

$$= \frac{1}{5} u^5 - \frac{2}{7} u^7 + \frac{1}{9} u^9 + C$$

$$= \frac{1}{5} \sin^5 x - \frac{2}{7} \sin^7 x + \frac{1}{9} \sin^9 x + C$$

Q Evaluate $\int \sin^4 x \cos^4 x \, dx$

m and n both are even

$$\int \sin^4 x \cos^4 x \, dx = \int (\sin^2 x)^2 (\cos^2 x)^2 \, dx$$

$$= \int \left[\frac{1}{2}(1-\cos 2x)\right]^2 \left[\frac{1}{2}(1+\cos 2x)\right]^2 dx$$

$$= \frac{1}{16} \int (1-\cos 2x)^2 (1+\cos 2x)^2 \, dx$$

$$= \frac{1}{16} \int (1-\cos^2 2x)^2 \, dx \qquad (\text{using } a^2-b^2 = (a-b)(a+b))$$

$$= \frac{1}{16} \int \sin^4 2x \, dx$$

let $u = 2x \Rightarrow du = 2dx \Rightarrow dx = \frac{1}{2} du$

$$\int \sin^4 x \cos^4 x \, dx = \frac{1}{32} \int \sin^4 u \, du$$

$$= \frac{1}{32} \left[\frac{3}{8} u - \frac{1}{4} \sin 2u + \frac{1}{32} \sin 4u\right] + C \quad \leftarrow (\text{Using reduction formula})$$

$$= \frac{3}{128} x - \frac{1}{128} \sin 4x + \frac{1}{1024} \sin 8x + C.$$

Q show that

$$\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx$$

Solution: $\int \tan^n x \, dx = \int \tan^{n-2} x \tan^2 x \, dx$

$$= \int \tan^{n-2} x (\sec^2 x - 1) \, dx \qquad (\text{using } \tan^2 x = \sec^2 x - 1)$$

$$= \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx$$

Now for the first integral on the right hand side

take $u = \tan x$ then $du = \sec^2 x \, dx$

$$\int \tan^{n-2} x \sec^2 x \, dx = \int u^{n-2} du = \frac{u^{n-1}}{n-1} = \frac{\tan^{(n-1)} x}{n-1}$$

Thus,

$$\int \tan^n x\,dx = \frac{\tan^{(n-1)} x}{(n-1)} - \int \tan^{(n-2)} x\,dx.$$

Q Show that

$$\int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{(n-1)} + \frac{(n-2)}{(n-1)} \int \sec^{n-2} x\,dx.$$

<u>Solution</u> Let $I_n = \int \sec^n x\,dx$.

Then $I_n = \int \sec^{n-2} x \sec^2 x\,dx$

Let $u = \sec^{n-2} x \Rightarrow du = (n-2)\sec^{n-3} x \sec x \tan x\,dx$

& let $v = \tan x \Rightarrow dv = \sec^2 x\,dx$

Now, $I_n = \int \sec^n x\,dx = \int u\,dv = uv - \int v\,du$

$= \sec^{n-2} x \tan x - (n-2)\int \tan x . \sec^{n-3} x \sec x \tan x\,dx$

$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x\,dx$

$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x (\sec^2 x - 1)\,dx$

$= \sec^{n-2} x \tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx$

$= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$

$\Rightarrow [1+(n-2)] I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$

$$\Rightarrow \int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{(n-1)} + \frac{(n-2)}{(n-1)} \int \sec^{n-2} x\,dx.$$

<u>Hence, Proved.</u>