Cylindrical Surfaces

The distance d between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is given by

$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

Example: Find the distance between the points $(2,3,-1)$ and $(4,-1,3)$.

Using the above formula,

$$d = \sqrt{(4-2)^2 + (-1-3)^2 + (3+1)^2}$$
$$= \sqrt{36}$$
$$= 6$$

Spheres

We recall that the equation in standard form of a circle in 2-space with centre (x_0, y_0) and radius r is

$$(x-x_0)^2 + (y-y_0)^2 = r^2.$$

This follows from the distance formula and the fact that the circle consists of all those points in 2-space whose distance from (x_0, y_0) is r.

Analogous to this, the standard equation of the sphere in 3-space that has center (x_0, y_0, z_0) and radius r is

$$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2.$$

This ~~formula~~ follows from distance formula and the fact that the sphere consists of all points in 3-space whose distance from (x_0, y_0, z_0) is r.

Example: $(x-3)^2 + (y-2)^2 + (z-1)^2 = 9$ represents equation of the sphere with center $(3,2,1)$ and radius 3.

Q Find the center and radius of the sphere

$$x^2+y^2+z^2-2x-4y+8z+17=0$$

Solution

We first write the given equation in the standard form by completing the squares.

$$(x^2-2x)+(y^2-4y)+(z^2+8z)=-17$$

$$\Rightarrow (x^2-2x+1)+(y^2-4y+4)+(z^2+8z+16) = -17+21$$

$$\Rightarrow (x-1)^2+(y-2)^2+(z+4)^2=4$$

which is the equation of the sphere with center $(1,2,-4)$ and radius 2.

Theorem: An equation of the form

$$x^2+y^2+z^2+Gx+Hy+Iz+J=0 \quad \text{—— (A)}$$

represents a sphere, a point or has no graph.

Using the last question it is clear that an equation of the form (A) can be converted into form of standard equation ~~of the sphere~~

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=K$$

by completing the squares.

i) If $K>0$, then the graph of this equation is a sphere with center (x_0,y_0,z_0) and radius $\sqrt{K}$.

ii) If $K=0$, then the sphere has radius zero, so the graph is the single point (x_0,y_0,z_0).

iii) If $K<0$, the equation is not satisfied by any values of x, y and z, so it has no graph.

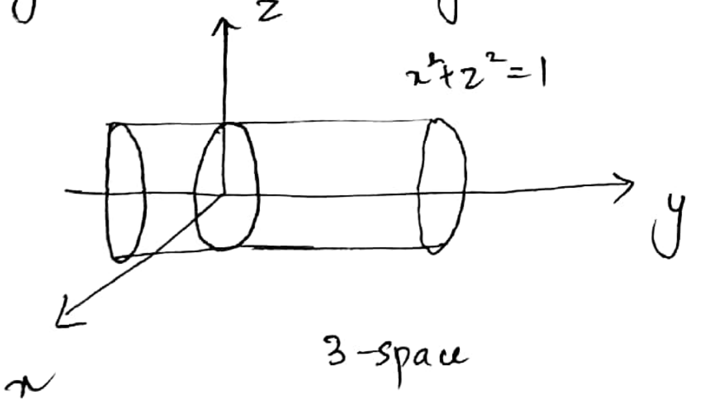
Cylindrical surfaces

It is natural to graph equations in two variables in 2-space and equations in three variables in 3-space, it is also possible to graph equations in two variables in 3-space. For example, the graph of the equation $y = x^2$ in an xy- coordinate system is a parabola. To obtain its graph in xyz-coordinate system we only need to Observe that the equation $y = x^2$ does not impose any restrictions on z. Thus, if we find values of x and y that satisfy this equation, then the coordinates of the point (x, y, z) will also satisfy the equation for arbitrary values of z.

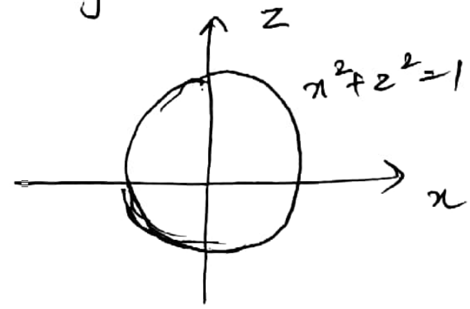
Geometrically, the point (x, y, z) lies on the vertical line through the point $(x, y, 0)$ in the xy-plane, which means that we can obtain the graph of $y = x^2$ in xyz- coordinate system by first graphing the equation in the xy-plane and then translating that graph parallel to the z-axis to generate the entire graph.

Theorem: An equation that contains only two of the variables x, y and z represents a cylindrical surface in an xyz- coordinate system. The surface can be obtained by graphing the equation in the coordinate plane of the two variables that appear in the equation and then translating that graph parallel to the axis of the missing variable.

Example: Sketch the graph of $x^2+z^2=1$ in 3-space.
Since y is missing in the equation. Therefore, using the above theorem we will first draw the graph of equation $x^2+z^2=1$ in xz-plane which is a circle. Thus, in 3-space the graph will be a right circular cylinder along the y-axis



3-space



2-space

Quadric Surfaces

In 3-space the shape of a surface is obtained by using network of mesh lines, which are curves obtained by cutting the surface with well-chosen planes. This is unlike 2-space, where shape of a curve can be obtained by plotting points, this method is not helpful in 3-space because too many points are required.

The mesh line that results when a surface is cut by a plane is called a trace of the surface in the plane. For example, consider the surface $z=x^2+y^2$. Its trace in the plane $z=K$ can be obtained by substituting $z=K$ in $z=x^2+y^2$, which gives

$$x^2+y^2=K \quad (z=k).$$

General second-order equation in xyz- coordinate system is (5)

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0.$$

The graph of such equations are called quadric surfaces or quadrics. Six common types of quadric surfaces are ellipsoids, hyperboloids of one sheet, hyperboloids of two sheets, elliptic cones, elliptic paraboloids and hyper paraboloids. Refer to table 11.7.1

Table 11.7.1

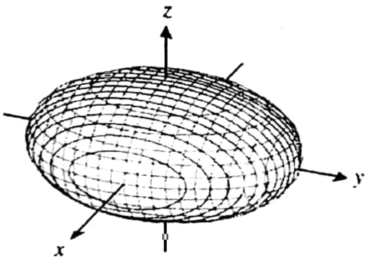
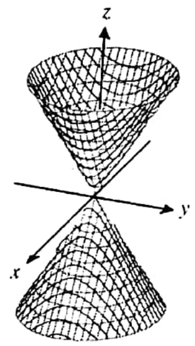
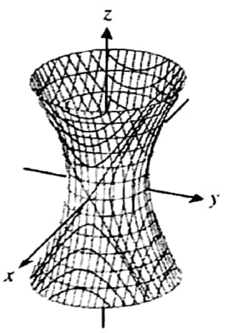
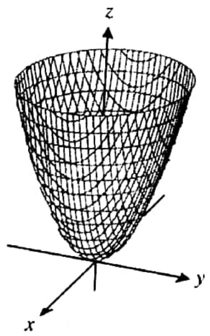
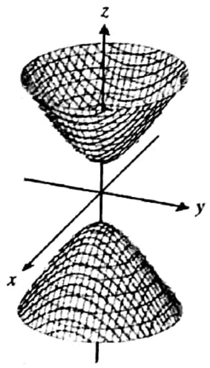
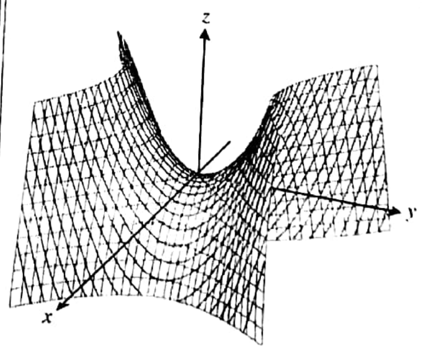
SURFACE	EQUATION	SURFACE	EQUATION
ELLIPSOID 	$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$ The traces in the coordinate planes are ellipses, as are the traces in those planes that are parallel to the coordinate planes and intersect the surface in more than one point.	**ELLIPTIC CONE** 	$$z^2=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$ The trace in the xy-plane is a point (the origin), and the traces in planes parallel to the xy-plane are ellipses. The traces in the yz- and xz-planes are pairs of lines intersecting at the origin. The traces in planes parallel to these are hyperbolas.
HYPERBOLOID OF ONE SHEET 	$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$ The trace in the xy-plane is an ellipse, as are the traces in planes parallel to the xy-plane. The traces in the yz-plane and xz-plane are hyperbolas, as are the traces in those planes that are parallel to these and do not pass through the x- or y-intercepts. At these intercepts the traces are pairs of intersecting lines.	**ELLIPTIC PARABOLOID** 	$$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$ The trace in the xy-plane is a point (the origin), and the traces in planes parallel to and above the xy-plane are ellipses. The traces in the yz- and xz-planes are parabolas, as are the traces in planes parallel to these.
HYPERBOLOID OF TWO SHEETS 	$$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$ There is no trace in the xy-plane. In planes parallel to the xy-plane that intersect the surface in more than one point the traces are ellipses. In the yz- and xz-planes, the traces are hyperbolas, as are the traces in those planes that are parallel to these.	**HYPERBOLIC PARABOLOID** 	$$z=\frac{y^2}{b^2}-\frac{x^2}{a^2}$$ The trace in the xy-plane is a pair of lines intersecting at the origin. The traces in planes parallel to the xy-plane are hyperbolas. The hyperbolas above the xy-plane open in the y-direction, and those below in the x-direction. The traces in the yz- and xz-planes are parabolas, as are the traces in planes parallel to these.

Table 11.7.2

IDENTIFYING A QUADRIC SURFACE FROM THE FORM OF ITS EQUATION

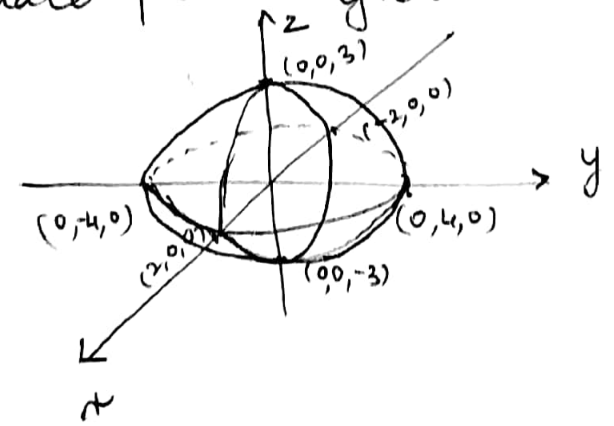
EQUATION	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$z^2-\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$	$z-\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$	$z-\frac{y^2}{b^2}+\frac{x^2}{a^2}=0$
CHARACTERISTIC	No minus signs	One minus sign	Two minus signs	No linear terms	One linear term; two quadratic terms with the same sign	One linear term; two quadratic terms with opposite signs
CLASSIFICATION	Ellipsoid	Hyperboloid of one sheet	Hyperboloid of two sheets	Elliptic cone	Elliptic paraboloid	Hyperbolic paraboloid

A rough sketch of an ellipsoid can be obtained by first plotting the intersections with coordinate axes & then sketching the elliptical traces in the coordinate planes.

Example: Sketch the ellipsoid

$$\frac{x^2}{4}+\frac{y^2}{16}+\frac{z^2}{9}=1$$

The x-intercepts can be obtained by setting $y=0$ & $z=0$ which gives $x=\pm 2$. Similarly, y-intercepts are $y=\pm 4$ and z-intercepts are $z=\pm 3$. Sketching the elliptical traces in coordinate planes yields the graph.

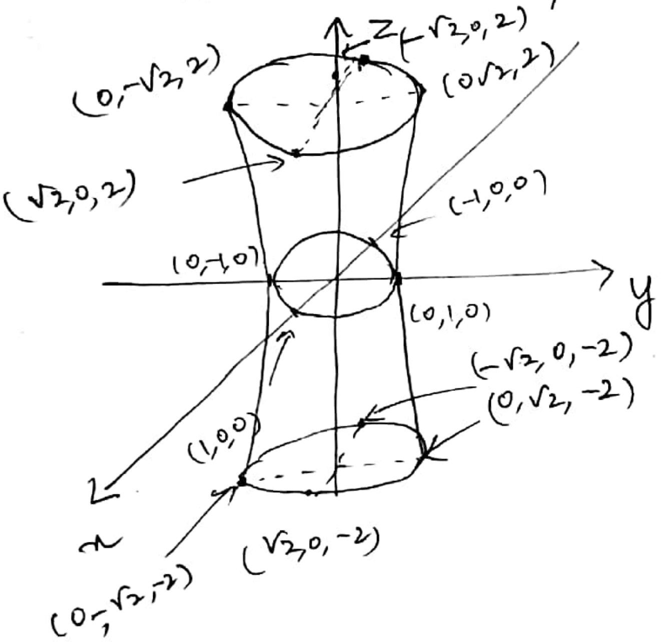


~~Example~~ A rough sketch of a hyperboloid of one sheet can be obtained by first sketching the elliptical trace in xy-plane, then the elliptical traces in the planes $z=\pm c$ & then the hyperbolic curves that join the endpoints of the axes of these ellipses.

Example: Sketch the graph of the hyperboloid of one sheet

$$x^2+y^2-\frac{z^2}{4}=1.$$

Solution The trace in the xy-plane, obtained by setting $z=0$ is $x^2+y^2=1$ $(z=0)$ which is a circle of radius 1 and center on z-axis. $(x=0, y=0)$ The traces in the planes $z=\pm 2$ are $x^2+y^2=2$ $(z=\pm 2)$ which are circles of radius $\sqrt{2}$ centered on z-axis $(x=y=0)$. Joining these circles by the hyperbolic traces in the vertical coordinate planes gives the graph



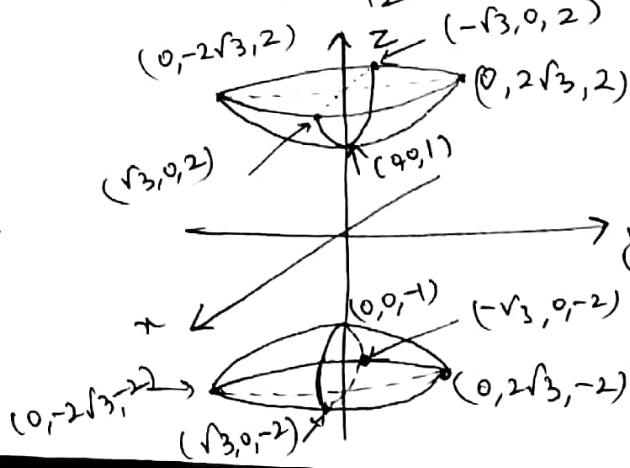
A rough sketch of the hyperboloid of two sheets can be obtained by first plotting the intersections with z-axis, then sketching the elliptical traces in the planes $z=\pm 2c$ & then sketching the hyperbolic traces that connect z-axis intersections & endpoints of the axes of the ellipses.

Example: Sketch the graph of the hyperboloid of two sheets

$$z^2-x^2-\frac{y^2}{4}=1.$$

Solution: The z-intercepts are $z=\pm 1$. The traces in the planes $z=2$ and $z=-2$ are $\frac{x^2}{3}+\frac{y^2}{12}=1$ $(z=\pm 2)$

Sketching these ellipses & the hyperbolic traces in the vertical coordinate planes gives the graph.



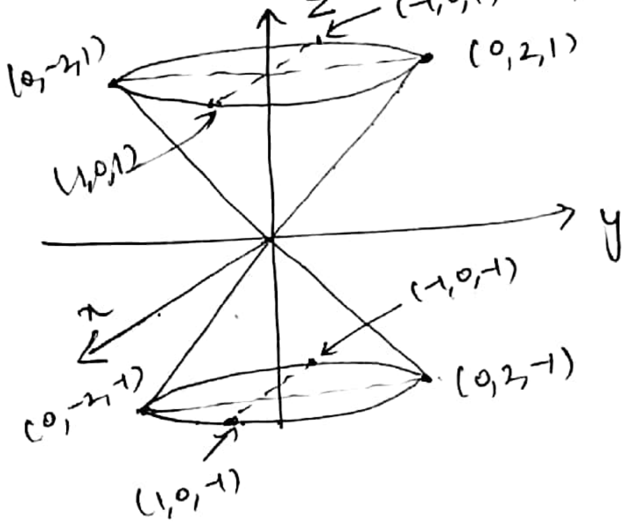
A rough sketch of the elliptic cone can be obtained by first sketching the elliptical traces in planes $z = \pm 1$ & then sketching the linear traces that connect the endpoints of the axes of the ellipses.

Q Sketch the graph of the elliptic cone $z^2 = x^2 + \frac{y^2}{4}$.

Solution: The traces in the planes $z = \pm 1$ are

$$x^2 + \frac{y^2}{4} = 1 \quad (z = \pm 1)$$

Sketching these ellipses & the linear traces in the vertical coordinate planes gives the graph.



A rough sketch of the elliptic paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \quad (a>0,\ b>0)$$

can be obtained by first sketching the elliptical trace in the plane $z = 1$ & then sketching the parabolic traces in the vertical coordinate planes to connect the origin to the ends of the axes of ellipse.

Q Sketch the graph of elliptic paraboloid $z = \frac{x^2}{4} + \frac{y^2}{9}$

Solution: The trace in the plane $z = 1$ is $\frac{x^2}{4} + \frac{y^2}{9} = 1 \ (z = 1)$

Sketching the ellipse & the parabolic traces in the vertical coordinate planes gives the graph.