Derivative of a vector-valued function

Definition: If $r(t)$ is a vector-valued function, then the derivative of r with respect to t is a vector-valued function r' defined as

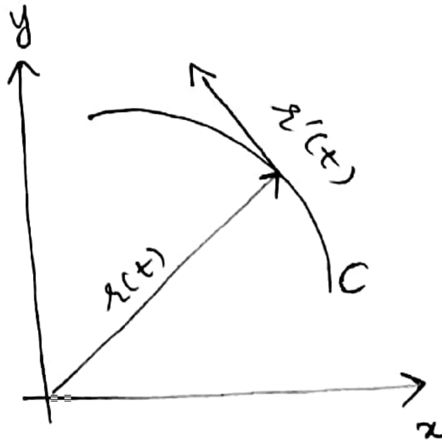
$$r'(t) = \lim_{h \to 0} \frac{r(t+h) - r(t)}{h}$$

The domain of r' is the set of all values of t in the domain of $r(t)$ for which the above limit exists. The function $r(t)$ is differentiable at t if the above limit exists.

We also denote the derivative of $r(t)$ by

$\frac{d}{dt}[r(t)]$, $\frac{dr}{dt}$, $r'(t)$ and r'.

Geometric interpretation of the derivative



Suppose that C is the graph of a vector-valued function $r(t)$ in 2-space or 3-space and that $r'(t)$ exists and is nonzero for a given value of t. If the vector $r'(t)$ is positioned with its initial point at the terminal point of the radius vector $r(t)$, then $r'(t)$ is tangent to C and points in the direction of increasing parameter.

Theorem (Only Statement) If $r(t)$ is a vector-valued function, then r is differentiable at t if and only if each of its component functions is differentiable at t, in which case the component functions of $r'(t)$ are the derivatives of the corresponding component functions of $r(t)$.

Example: Let $r(t) = t^2\hat{i} + e^t\hat{j} - (2\cos\pi t)\hat{k}$.

Then $r'(t) = \frac{d}{dt}(t^2)\hat{i} + \frac{d}{dt}(e^t)\hat{j} - \frac{d}{dt}(2\cos\pi t)\hat{k}$

$= 2t\,\hat{i} + e^t\hat{j} + (2\pi\sin\pi t)\hat{k}$

Theorem: (Rules of Differentiation) Let $r(t)$, $r_1(t)$ & $r_2(t)$ be differentiable vector-valued function that are all in 2-space or all in 3-space, and let $f(t)$ be a differentiable real-valued function, k be a scalar, and c be a constant vector. Then, the following rules of differentiation hold:

(i) $\frac{d}{dt}[c] = 0$

(ii) $\frac{d}{dt}[k\,r(t)] = k\frac{d}{dt}[r(t)]$

(iii) $\frac{d}{dt}[r_1(t) + r_2(t)] = \frac{d}{dt}[r_1(t)] + \frac{d}{dt}[r_2(t)]$

(iv) $\frac{d}{dt}[r_1(t) - r_2(t)] = \frac{d}{dt}[r_1(t)] - \frac{d}{dt}[r_2(t)]$

(v) $\frac{d}{dt}[f(t)\,r(t)] = f(t)\frac{d}{dt}[r(t)] + \frac{d}{dt}[f(t)]\,r(t)$

Q Let $r(t) = t^2 u(t)$ where $u(t) = t\hat{i} + t^2\hat{j} - t^3\hat{k}$.
Find $r'(t)$.

Solution $r'(t) = \frac{d}{dt}[t^2 u(t)] = \frac{d}{dt}(t^2)\, u(t) + t^2 \frac{d}{dt}(u(t))$

[using part (v) of previous theorem]

$= 2t(t\hat{i} + t^2\hat{j} - t^3\hat{k}) + t^2(\hat{i} + 2t\hat{j} - 3t^2\hat{k})$

$= 3t^2\hat{i} + 4t^3\hat{j} - 5t^4\hat{k}$.

Derivatives of dot and Cross products

$$\frac{d}{dt}[r_1(t) \cdot r_2(t)] = r_1(t) \cdot \frac{dr_2}{dt} + \frac{dr_1}{dt} \cdot r_2(t)$$

$$\frac{d}{dt}[r_1(t) \times r_2(t)] = r_1(t) \times \frac{dr_2}{dt} + \frac{dr_1}{dt} \times r_2(t).$$

Q If $r_1(t) = t\hat{i} + t^2\hat{j} - t^3\hat{k}$ and $r_2(t) = \sin t\,\hat{i} + 2\cos t\,\hat{j} + \cos t\,\hat{k}$
Find $\frac{d}{dt}[r_1(t) \cdot r_2(t)]$ and $\frac{d}{dt}[r_1(t) \times r_2(t)]$.

Solution $\frac{d}{dt}[r_1(t) \cdot r_2(t)] = r_1(t) \cdot r_2'(t) + r_1'(t)\, r_2(t)$

$= (t\hat{i} + t^2\hat{j} - t^3\hat{k})(\cos t\,\hat{i} - 2\sin t\,\hat{j} - \sin t\,\hat{k}) + (\hat{i} + 2t\hat{j} - 3t^2\hat{k})(\sin t\,\hat{i} + 2\cos t\,\hat{j} + \cos t\,\hat{k})$

$= (1 - 2t^2 + t^3)\sin t + (5t - 3t^2)\cos t$

Now $\frac{d}{dt}[r_1(t) \times r_2(t)] = r_1(t) \times r_2'(t) + r_1'(t) \times r_2(t)$

$= (t\hat{i} + t^2\hat{j} - t^3\hat{k}) \times (\cos t\,\hat{i} - 2\sin t\,\hat{j} - \sin t\,\hat{k}) + (\hat{i} + 2t\hat{j} - 3t^2\hat{k}) \times (\sin t\,\hat{i} + 2\cos t\,\hat{j} + \cos t\,\hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t & t^2 & -t^3 \\ \cos t & -2\sin t & -\sin t \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2t & -3t^2 \\ \sin t & 2\cos t & \cos t \end{vmatrix}$$

$= \{2t(1+3t)\cos t - t^2(1+2t)\sin t\}\hat{i} + [-(1+t^3)\cos t + t(1-3t)\sin t]\hat{j}$
$+ [(2-t^2)\cos t - 4t\sin t]\hat{k}$

If ϕ is a function of three variables, then the gradient of ϕ is defined as (Scalar $\rightarrow$ Vector)

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

<u>Example</u>: let $\phi(x,y) = x+y$. Then

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} = \hat{i}+\hat{j}$$

<u>Definition</u>: If $F(x,y,z) = f(x,y,z)\hat{i} + g(x,y,z)\hat{j} + h(x,y,z)\hat{k}$, then we define the divergence of F, denoted by div F, (Vector $\rightarrow$ Scalar)

as
$$\text{div } F = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

<u>Definition</u>: If $F(x,y,z) = f(x,y,z)\hat{i} + g(x,y,z)\hat{j} + h(x,y,z)\hat{k}$, denoted by curl F, then we define the curl of F, (Vector $\rightarrow$ vector)

as
$$\text{curl } F = \nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\hat{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\hat{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\hat{k}$$

In the study of fluid flow, divergence relates to the way in which fluid flows towards or away from a point and curl relates to the rotational properties of the fluid at a point.

Example: Find the divergence & the curl of the vector field $F(x,y,z) = x^2y\,\hat{i} + 2y^3z\,\hat{j} + 3z\,\hat{k}$.

$$\text{div } F = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(2y^3z) + \frac{\partial}{\partial z}(3z)$$

$$= 2xy + 6y^2z + 3$$

and

$$\text{curl } F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & 2y^3z & 3z \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(3z) - \frac{\partial}{\partial z}(2y^3z)\right]\hat{i} + \left[\frac{\partial}{\partial z}(x^2y) - \frac{\partial}{\partial x}(3z)\right]\hat{j}$$

$$+ \left[\frac{\partial}{\partial x}(2y^3z) - \frac{\partial}{\partial y}(x^2y)\right]\hat{k}$$

$$= -2y^3\,\hat{i} - x^2\,\hat{k}$$