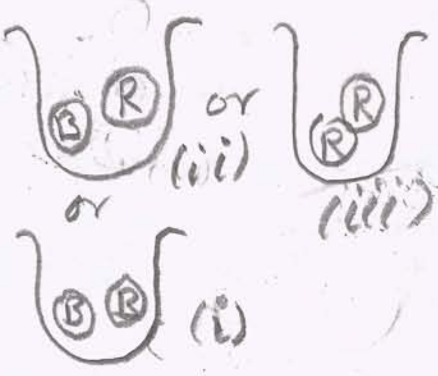
## Example 4.10

An urn always contains 2 balls.




A ball is randomly chosen and then replaced by a new ball, and

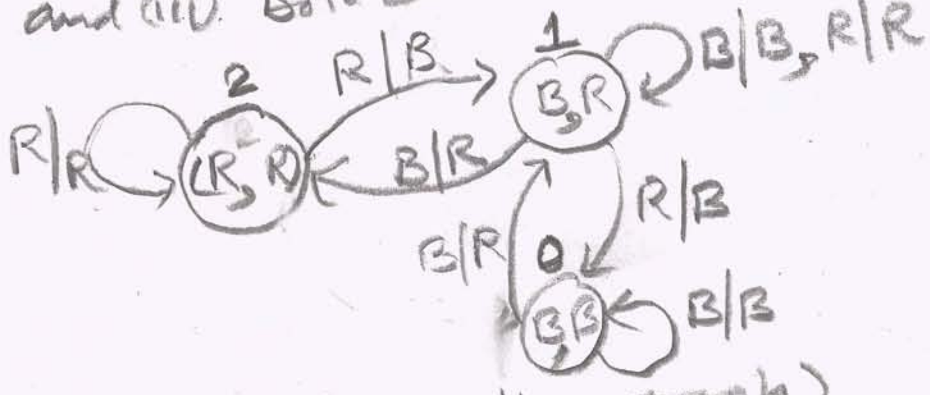
P[ Drawn ball & Replaced ball have the same color ] = 0.8;

P[ Drawn ball & Replaced ball are of the opposite color ] = 0.2.

Question - If initially both balls are red (ii), find the probability that fifth ball selected is red.

Solution. Let $X_n$ be the number of red balls in the urn after the nth selection and subsequent replacement.

As shown above, there are 3 possible states at any instance, (i) Both balls are of blue (state 0) color, (ii) Balls are of opposite colors (state 1) and (iii) Both ball are red (initial state, 2)




(Transition graph)

We have $X_0 = 2$.

Continued . .

$p_{20} = P[X_1 = 0 | X_0 = 2] = 0$; (No transition from state 2 to state 0)

$p_{21} = P[X_1 = 1 | X_0 = 2] = 0.2$;

$p_{22} = P[X_1 = 2 | X_0 = 2] = 0.8$;

$p_{10} = P[X_{n+1} = 0 | X_n = 1] = \frac{1}{2} \times 0.2 = 0.1$

$p_{11} = P[X_{n+1} = 1 | X_n = 1] = 0.8$,

$p_{12} = P[X_{n+1} = 2 | X_n = 1] = \frac{1}{2} \times 0.2 = 0.1$

Note that, $P[X_{n+1} = j | X_n = i] = P[X_1 = j | X_0 = i]$

$p_{00} = P[X_{n+1} = 0 | X_n = 0] = 0.8$;

$p_{01} = P[X_{n+1} = 1 | X_n = 0] = 0.2 \times 1 = 0.2$

$p_{02} = P[X_{n+1} = 2 | X_n = 0] = 0$;

Thus, $X_n, n \geq 0$ is a Markov Chain with states $\{0, 1, 2\}$ having transition probability matrix $P$ given by

$$P = \begin{Vmatrix} 0.8 & 0.2 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.2 & 0.8 \end{Vmatrix}$$

To determine the probability that the fifth selection is red, depends on the number of red balls in the urn after the 4th selection. Let $A$ denotes the "Selection of red ball in 5th trial".

Then, $P[A] = P[A | X_4 = 0] \times P[X_4 = 0 | X_0 = 2]$
$\quad + P[A | X_4 = 1] \times P[X_4 = 1 | X_0 = 2]$
$\quad + P[A | X_4 = 2] \times P[X_4 = 2 | X_0 = 2]$

Continued

$\therefore$ P[ fifth selection is red, A]

$$= \sum_{i=0}^{2} P[A \mid X_4 = i]\, P[X_4 = i \mid X_0 = 2]$$

$$= 0 \times P_{2,0}^{(4)} + (0.5)\, P_{2,1}^{(4)} + 1\, P_{2,2}^{(4)}$$

where $P_{2,0}^{(4)} = P[X_{4+k} = 0 \mid X_k = 2]$;

$P_{2,1}^{(4)} = P[X_{4+k} = 1 \mid X_k = 2]$;

$P_{2,2}^{(4)} = P[X_{4+k} = 2 \mid X_k = 2]$.

Now, we know that $P^{(4)}$ denote the matrix of 4-step transition probabilities $P_{ij}^{(4)}$: we calculate $P^{(4)}$ as follows:

$$P^{(2)} = P^{(1)} P^{(1)} = P\,P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ 0 & 0.8 & 0.2 & 0 \\ 1 & 0.1 & 0.8 & 0.1 \\ 2 & 0 & 0.2 & 0.8 \end{array} \begin{Vmatrix} 0.8 & 0.2 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.2 & 0.8 \end{Vmatrix}$$

$$= \begin{array}{c} 0 \\ 1 \\ 2 \end{array} \begin{Vmatrix} 0.66 & 0.32 & 0.02 \\ 0.16 & 0.68 & 0.16 \\ 0.02 & 0.32 & 0.66 \end{Vmatrix}, \text{ and}$$

$$P^{(4)} = P^{(2)} P^{(2)} = \begin{array}{c} 0 \\ 1 \\ 2 \end{array} \begin{Vmatrix} - & - & - \\ - & - & - \\ 0.0776 & 0.4352 & 0.4872 \end{Vmatrix}$$

So, we have

$P_{2,0}^{(4)} = 0.0776$; $P_{2,1}^{(4)} = 0.4352$; $P_{2,2}^{(4)} = 0.4872$

Therefore, p(fifth selection is red)

$$= 0 \times 0.0776 + (0.5) \times 0.4352 + 0.4872$$

$$= 0.7048 \quad //$$