3.2 COMPARISON TESTS FOR POSITIVE TERM SERIES

For a given series Σa_n, there are mainly two types of problems to investigate - first, whether Σa_n is convergent. Second, if Σa_n is convergent, what is its sum ? There are several results to help us deal with the first type of problems. These results are known as 'tests' for convergence of series. The second problem is quite difficult at times. In fact there are many examples of convergent series whose exact sum is not known.

If the terms of a series Σa_n are non-negative, then the associated sequence of partial sums (S_n) is an increasing sequence. This can be easily verified as $S_{n+1} - S_n = a_{n+1} \geq 0$.

As a consequence, the sequence of partial sums converges if it is bounded, in view of the Monotone Convergence Theorem. We shall use this fact in the following proofs.

Theorem (Comparison Test) 3.2.1. Let $0 \leq a_n \leq b_n$ for all $n \in \mathbf{N}$.

(a) If Σb_n converges, then Σa_n converges

(b) If Σa_n diverges, then Σb_n diverges.

Proof. (a) We write $S_n = a_1 + a_2 + \ldots + a_n$, $T_n = b_1 + b_2 + \ldots + b_n$.

Then $S_n \leq T_n$ for all $n \in \mathbf{N}$. As Σb_n converges, (T_n) is convergent.

Let $\lim(T_n) = T$. Since, (T_n) is an increasing sequence, we have

$$T = \sup\{T_n : n \in \mathbf{N}\}$$

by the Monotone Convergence Theorem.

Hence, $T_n \leq T$ for all $n \in \mathbf{N}$.

As a result we have,

$$S_n \leq T_n \leq T \quad \text{for all } n \in \mathbf{N}.$$

Thus, (S_n) is an increasing sequence which is bounded above.

Therefore (S_n) is convergent. That is, Σa_n converges.

(b) It follows from (a), because if Σb_n converges then Σa_n must converge by (a), giving us a contradiction.

Example 3.2.2. (a) The series $\sum_{n=1}^{\infty} \frac{n+2}{n3^n}$ has its n^{th} term $a_n = \frac{n+2}{n.3^n}$.

Now, $\frac{n+2}{n} = 1 + \frac{2}{n} \leq 3$ for all $n \in \mathbf{N}$.

Hence,

$$a_n \le \frac{3}{3^n} \qquad \text{for all } n \in \mathbf{N}$$

Since the geometric series $\Sigma \frac{1}{3^{n-1}}$ is convergent (as $r = \frac{1}{3} < 1$ here), the given series is convergent, by the comparison test.

(b) The series $\sum_{n=1}^{\infty} e^n$ is divergent as $e^n \ge 2^n$ for all $n \in \mathbf{N}$ and the series $\Sigma 2^n$ is divergent with r = 2.

(c) $\Sigma \frac{1}{n!} = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + ...$ is convergent.

Let $a_n = \frac{1}{n!}$, $b_n = \frac{1}{2^{n-1}}$.

Since $n! \ge 2^{n-1}$ for all $n \in \mathbf{N}$.

We have,

$$a_n \le b_n \qquad \text{for all } n \in \mathbf{N}$$

Now, the series $\Sigma \frac{1}{2^{n-1}}$ converges as $r = \frac{1}{2}$ here.

Hence, by the comparison test, $\Sigma \frac{1}{n!}$ is convergent.

Theorem (Limit Comparison Test) 3.2.3. Let Σa_n and Σb_n be two series of positive terms. If the sequence $\left(\frac{a_n}{b_n}\right)$ converges to a finite positive limit, then both Σa_n and Σb_n converge or diverge together.

Proof. Suppose that $\lim\left(\frac{a_n}{b_n}\right) = L > 0$. Then for $\varepsilon = \frac{L}{2}$, there exists $m \in \mathbf{N}$ such that

$$\left|\frac{a_n}{b_n} - L\right| < \frac{L}{2} \qquad \text{for all } n \ge m$$

Thus, $L - \frac{L}{2} < \frac{a_n}{b_n} < L + \frac{L}{2}$ for all $n \ge m$

$$\Rightarrow \qquad \frac{L}{2} b_n < a_n < \frac{3L}{2} b_n \qquad \text{for all } n \geq m \qquad \text{(i)}$$

Now, if $\sum_{n=1}^{\infty} a_n$ converges, so does $\sum_{n=m}^{\infty} a_n$ by Theorem 3.1.9.

Then by the comparison test, $\sum_{n=m}^{\infty} \frac{L}{2} b_n$ converges in view of the first part of inequality of (i).

This implies that $\sum_{n=m}^{\infty} b_n$ and hence $\sum_{n=1}^{\infty} b_n$ converge, again by Theorem 3.1.9.

If $\sum_{n=1}^{\infty} b_n$ converges, we use the second part of the inequality of (i) and argue similarly.

Similarly, divergence of $\sum_{n=1}^{\infty} a_n$ implies divergence of $\sum_{n=1}^{\infty} b_n$ and vice–versa.

Corollary 3.2.4. Let Σa_n and Σb_n be positive term series. If $\lim \left(\frac{a_n}{b_n} \right) = 0$ and Σb_n is convergent, then Σa_n is convergent.

Proof. We take $\varepsilon = 1$. Then there exists $m \in \mathbf{N}$ such that

$$\left| \frac{a_n}{b_n} - 0 \right| < 1 \qquad \text{for all } n \geq m$$

$$\Rightarrow \qquad a_n < b_n \qquad \text{for all } n \geq m$$

Then following the argument of Theorem 3.2.3, we find that Σa_n is convergent.

Remark. For two positive term series Σa_n and Σb_n, if $\lim \left(\frac{a_n}{b_n} \right) = \infty$, then above corollary says that if Σa_n converges, so does Σb_n. But what can you say if $\lim \left(\frac{a_n}{b_n} \right) = 0$ and Σa_n converges ?

Example 3.2.5. Test the convergence of the series $\sum \frac{1}{3^n + x}$, $x > 0$

Solution. Here $a_n = \dfrac{1}{3^n + x}$

We take $b_n = \dfrac{1}{3^n}$.

Then $\lim\left(\dfrac{a_n}{b_n}\right) = \lim\left(\dfrac{3^n}{3^n + x}\right) = 1$

Therefore, by the limit comparison test, Σa_n converges as $\Sigma \dfrac{1}{b_n} = \Sigma \dfrac{1}{3^n}$ is a geometric series with $r = \dfrac{1}{3}$ and hence is convergent.

In the above examples, while applying the comparison test, we have used the geometric series as our 'testing series' . However, any series, whose behaviour is known, can be used for this purpose. One such series is $\Sigma \dfrac{1}{n^p}$, known as p-series. Below we provide a result stating the behaviour of p-series, so that we can use it in the comparison tests.

Theorem 3.2.6. The series $\sum_{n=1}^{\infty} \dfrac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

We shall provide a simple proof of this result later in Section 3.4 by using Cauchy's Integral test (see Example 3.4.2). However we use the result here as a tool for comparison test.

For $p = 1$, we get

$$\Sigma \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots$$

This series is known as '*harmonic series*'. It is divergent as $p = 1$ here.

For $p = 2$, we get $\sum \dfrac{1}{n^2} = 1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \ldots + \dfrac{1}{n^2} + \ldots$

which is convergent as $p = 2$ here.

You may notice here that bigger is the value of p, smaller is the value of $\dfrac{1}{n^p}$. In fact, with $p > 1$, the terms of $\Sigma \dfrac{1}{n^p}$ are small enough to give a finite sum.

Example 3.2.7. Test for convergence the series

$$\frac{1}{3}+\frac{\sqrt{2}}{5}+\frac{\sqrt{3}}{7}+\frac{\sqrt{4}}{9}+\ldots$$

Solution. Here $a_n = \dfrac{\sqrt{n}}{2n+1}$

For large value of n, a_n behaves as $\dfrac{\sqrt{n}}{2n}$, that is, as $\dfrac{1}{2\sqrt{n}}$.

Therefore, we take, $b_n = \dfrac{1}{\sqrt{n}}$

Then $\dfrac{a_n}{b_n} = \dfrac{\sqrt{n}}{2n+1} / \dfrac{1}{\sqrt{n}} = \dfrac{n}{2n+1}$

$$\lim\left(\frac{a_n}{b_n}\right) = \lim\left(\frac{n}{2n+1}\right) = \lim\left(\frac{1}{2+\frac{1}{n}}\right) = \frac{1}{2}$$

which is finite and positive.

Hence, by the limit comparison test, Σa_n and Σb_n converge or diverge together.

Since, $\Sigma\dfrac{1}{\sqrt{n}}$ is a divergent series, as $\Sigma\dfrac{1}{n^p}$ diverge for $p \le 1$, the given series is also divergent.

Example 3.2.8. Test for convergence the series $\Sigma(\sqrt{n^6+1}-n^3)$

Solution. Here $a_n = \sqrt{n^6+1}-n^3 = \dfrac{1}{\sqrt{n^6+1}+n^3}$ (on rationalizing)

For large values of n, a_n behaves as $\dfrac{1}{2n^3}$.

Hence, we take, $b_n = \dfrac{1}{n^3}$

Then $\dfrac{a_n}{b_n} = \dfrac{n^3}{\sqrt{n^6+1}+n^3} = \dfrac{1}{\sqrt{1+\dfrac{1}{n^6}}+1}$

$$\lim\left(\frac{a_n}{b_n}\right) = \lim\left(\frac{1}{\sqrt{1+\frac{1}{n^6}}+1}\right) = \frac{1}{2}$$

which is finite and positive.

Hence, by the limit comparison test Σa_n and Σb_n converge or diverge together.

However, $\Sigma \frac{1}{n^3}$ is convergent by the p–test as p = 3 here. Hence, the given series is convergent.

ILLUSTRATIVE EXAMPLES

For comparison of series, the following inequality is often helpful

$\ln n < n < e^n$ for all $n \in \mathbf{N}$.

1. Test for convergence the following series

(a) $\Sigma \tan^{-1}\left(\frac{1}{n}\right)$ (b) $\Sigma \frac{1}{n^n}$ (c) $\Sigma \sin\frac{1}{n}$

(d) $\Sigma \frac{\ln n}{n^2}$ (e) $\Sigma \frac{\ln n}{n^3}$ (f) $\Sigma \frac{\sqrt{n}}{\ln n}$

(g) $\Sigma \frac{1}{\ln n}$ (h) $\Sigma \frac{1}{2^n + n}$ (i) $\Sigma \frac{\ln n}{n!}$

(j) $\Sigma \frac{2+\sin n}{3^n}$ (k) $\Sigma \left[\frac{1}{n+(-1)^n}\right]^2$ (l) $\Sigma[\sqrt{n+1}-\sqrt{n}]$

(m) $\Sigma(\sqrt{n^4+1}-n^2)$

Solution. We write $a_n = \tan^{-1}\left(\frac{1}{n}\right)$ and take $b_n = \frac{1}{n}$

Then, $$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\frac{\tan^{-1}(1/n)}{1/n}$$

$$= \lim_{\theta\to 0}\frac{\tan^{-1}\theta}{\theta}, \quad \text{where } \theta = \frac{1}{n}$$

$$= 1$$

Now, the series Σb_n is divergent as $\Sigma \frac{1}{n^p}$ diverges for p = 1 .

Hence, by the limit comparison test, the series $\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{n}\right)$ diverges.

(b) Here $a_n = \frac{1}{n^n}$. For $n \geq 2$, $n^n \geq 2^n$. Hence

$$\frac{1}{n^n} \leq \frac{1}{2^n} \quad \text{for all } n \geq 2$$

Since $\Sigma \frac{1}{2^n}$ is a G.P. series with $|r| = \frac{1}{2} < 1$, it is convergent.

Hence, by the comparison test $\Sigma \frac{1}{n^n}$ is convergent.

(c) Here $a_n = \sin \frac{1}{n}$. We take $b_n = \frac{1}{n}$.

Then, $\lim \left(\frac{a_n}{b_n}\right) = 1$.

Therefore, by the limit comparison test $\Sigma \sin \frac{1}{n}$ diverges, as $\Sigma \frac{1}{n}$ diverges by the p-test.

(d) We have, $1 \leq \ln n \leq n$ for all $n \geq 3$.

Accordingly

$$\frac{1}{n^2} \leq \frac{\ln n}{n^2} \leq \frac{1}{n} \qquad \text{for all } n \geq 3.$$

Therefore, we choose, $b_n = \frac{1}{n^{3/2}}$. Then

$$\frac{a_n}{b_n} = \frac{\ln n}{n^2} . n^{3/2} = \frac{\ln n}{\sqrt{n}}$$

Here we apply L'Hospital's rule (please see Theorem 7.2.4, for details) in the following way

$$\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}} \left(\frac{\infty}{\infty}\right) = \lim_{x \to \infty} \frac{(1/x)}{\left(\frac{1}{2\sqrt{x}}\right)} = \lim_{x \to \infty} \left(\frac{2}{\sqrt{x}}\right) = 0$$

Hence, $\lim\left(\frac{a_n}{b_n}\right) = 0$.

Since Σb_n is convergent, as $\Sigma \frac{1}{n^p}$ converges for p > 1, $\Sigma \frac{\ln n}{n^2}$ is convergent by Corollary of the Comparison Test.

(e) Use the fact that $\frac{\ln n}{n^3} \le \frac{n}{n^3} = \frac{1}{n^2}$ for all $n \ge 1$.

(f) $a_n = \frac{\sqrt{n}}{\ln n}$

As $\ln n \le n$ for all n, we have

$$\frac{\sqrt{n}}{\ln n} \ge \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} \quad \text{for all n.}$$

Now, $\Sigma \frac{1}{\sqrt{n}}$ is divergent as $p = \frac{1}{2} < 1$.

Hence, the given series is divergent by the comparison test.

(g) Hint : $\ln n \le n$

(h) Hint : $\frac{1}{2^n + n} \le \frac{1}{2^n}$

(i) Hint : $\frac{\ln n}{n!} \le \frac{n}{n!} = \frac{1}{(n-1)!} \le \frac{1}{2^{n-2}}$

(j) Hint : $\frac{2 + \sin n}{3^n} \le \frac{3}{3^n}$

(k) Hint : $\frac{1}{[n + (-1)^n]^2} \le \frac{1}{(n-1)^2}$

(l) Hint : Rationalize, take $b_n = \frac{1}{\sqrt{n}}$

(m) Hint : Rationalize, take $b_n = \frac{1}{n^2}$

2. If Σa_n is a convergent positive term series and p > 1, then Σa_n^p is also convergent.

Solution. First we show that $a_n < 1$ ultimately.

Since, Σa_n is convergent, $\lim(a_n) = 0$.

Hence, for ε = 1, there exists m ∈ **N** such that

$$|a_n - 0| < 1 \text{ for all } n \geq m$$

That is,

$$a_n < 1 \qquad \text{for all } n \geq m$$

Now, for p > 1,

$$a_n^p \leq a_n \qquad \text{for all } n \geq m, \text{ as } a_n < 1.$$

Hence, by the comparison test, Σa_n^p is convergent.

3. Check whether the series

$$\frac{\sqrt{2}\ln 4}{1.2} + \frac{\sqrt{5}\ln 7}{2.3} + \frac{\sqrt{8}\ln 10}{3.4} + \frac{\sqrt{11}\ln 13}{4.5} + \ldots$$

converges or diverges.

Solution. Here $a_n = \frac{\sqrt{3n-1}\ln(3n+1)}{n(n+1)}$

On neglecting the finite quantities from a_n, we get

$$a_n \approx \frac{\sqrt{n}\ln n}{n^2} = \frac{\ln n}{n\sqrt{n}} = \frac{\ln n}{n^{3/2}}$$

Therefore, we take $b_n = \frac{\ln n}{n^{3/2}}$.

Then, $\frac{a_n}{b_n} = \frac{\sqrt{3n-1}\ln(3n+1)}{n(n+1)} \cdot \frac{n^{3/2}}{\ln n}$

$$= \frac{\sqrt{3 - \frac{1}{n}}}{1 + \frac{1}{n}} \left(\frac{\ln 3 + \ln(n + \frac{1}{3})}{\ln n} \right)$$

$$\lim \left(\frac{a_n}{b_n} \right) = \sqrt{3}$$

Hence, by the limit comparison test, Σa_n and Σb_n converge or diverge together. Since, $\sum \frac{\ln n}{n^p}$ converges for $p > 1$ (See Illustrative Example 3 of Section 3.4), the series

$\sum b_n = \sum \frac{\ln n}{n^{3/2}}$ is convergent.

Hence, Σa_n is convergent.

EXERCISES 3.2

1. Test for convergence the following series

(a) $\Sigma \sin\frac{1}{n}$ (b) $\Sigma \cot^{-1} n^2$ (c) $\Sigma \frac{1}{n}\sin\frac{1}{n}$

(d) $\Sigma \frac{1}{\sqrt{n}}\tan\frac{1}{n}$ (e) $\Sigma \sin^{-1}\frac{1}{n}$ (f) $\Sigma \sin\frac{1}{n^2}$

(g) $\Sigma \frac{1}{n}\tan^{-1}\frac{1}{n}$ (h) $\Sigma \frac{1}{\sqrt{n}}\sin\frac{1}{n}$ (i) $\Sigma \sin\frac{\pi}{2^n}$

(j) $\Sigma(a^{1/n}-1)$, $a>0$ (k) $\Sigma \frac{1}{n!}$ (l) $\Sigma \frac{1}{\ln n}$, $n \geq 2$

(m) $\Sigma \frac{1}{n^2 \ln n}$ (n) $\Sigma \frac{\sin(1/n!)}{\cos(1/n!)}$ (o) $\Sigma \frac{1}{3^n + x}$, $x > 0$

(p) $\Sigma \frac{1.3.5...(2n-1)}{2.4.6...2n}$ (q) $\Sigma\left(\sqrt{n^4+1}-\sqrt{n^4-1}\right)$ (r) $\Sigma \frac{1}{\sqrt{n^3+1}}$

(s) $\Sigma \frac{3+\cos n}{n^2-4}$ (t) $\Sigma \frac{1}{(\ln n+1)}$ (u) Σe^{-n^2}

(v) $\Sigma \frac{1}{n^{1+\frac{1}{n}}}$

2. Examine the convergence of the following series

(a) $1+\frac{1}{4^{2/3}}+\frac{1}{9^{2/3}}+\frac{1}{16^{2/3}}+...$ (b) $\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\frac{5}{4^p}+...$

(c) $\frac{2^p}{1^q}+\frac{3^p}{2^q}+\frac{4^p}{3^q}+...$, $p>0$, $q>0$ (d) $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+...$

(e) $\frac{1}{3^2}.\frac{2}{4^2}+\frac{3}{5^2}.\frac{4}{6^2}+\frac{5}{7^2}.\frac{6}{8^2}+...$ (f) $\frac{3}{5}+\frac{4}{5^2}+\frac{3}{5^3}+\frac{4}{5^4}+...$

(g) $a+b+a^2+b^2+a^3+b^3+....$

3. If Σa_n^2 converges, $a_n \geq 0$ for all n, then show that $\Sigma \frac{a_n}{n}$ also converges.

4. If Σa_n^2 and Σb_n^2 converge, $a_n \geq 0$, $b_n \geq 0$, then show that $\Sigma a_n b_n$ also converges.

5. Show that if Σa_n is convergent and $a_n \geq 0$ for all n, then each of the following series is convergent :

(a) Σa_n^2 (b) $\Sigma a_n a_{n+1}$ (c) $\Sigma(a_n - a_{n+1})$

(d) $\Sigma \frac{a_n}{1+a_n}$ (e) $\Sigma n^{1/n} a_n$ (f) $\Sigma \left(1+\frac{1}{n}\right)^n a_n$

6. Test for convergence

$$1+\frac{2}{5}x+\frac{6}{9}x^2+\frac{14}{17}x^3+\ldots+\frac{2^n-2}{2^n+1}x^{n-1}+\ldots,\ x>0$$

7. Examine for converge the series $\Sigma \frac{a^n}{a^n+x^n}$, $a > 0$, $x > 0$.

8. If $a_n \geq 0$, $b_n \geq 0$, then which of the following hold :

(a) Σa_n, Σb_n converge implies $\Sigma a_n b_n$ converges

(b) $\Sigma a_n, \Sigma a_n b_n$ converge implies Σb_n converges

(c) Σa_n converges, Σb_n diverges imply $\Sigma a_n b_n$ converges ?

9. If $0 \leq a_n \leq 1$ and $0 < x < 1$, then show that $\Sigma a_n x^n$ converges.

10. Test the convergence of the series

$$\frac{1}{1+x}+\frac{x}{1+x^2}+\frac{x^2}{1+x^3}+\ldots.,\ x>0$$

3.3 RATIO TEST AND ROOT TEST

In the comparison tests, we compare the given series with another series whose behaviour is known. Similarly in the integral test, we take the help of an integrable function which coincides with the terms of the given series for the integral values. In the following tests that we discuss now, no external help is required for determining the convergence of a given series. Out of these two tests, the root test is stronger than the ratio test in the sense that root test is applicable whenever the ratio test is applicable. The converse is, however, not true.

Before proving the ratio test for series, students are advised to revisit the previous chapter to have a look at the 'ratio test for sequences'.

Theorem (De Almbert's Ratio Test) 3.3.1. Let Σa_n be a positive term series and let $\lim\left(\frac{a_{n+1}}{a_n}\right) = l$. Then

(i) If $l < 1$, the series $\sum a_n$ converges;

(ii) If $l > 1$, the series diverges;

(iii) If $l = 1$, the test fails.

Proof.

Case-(i) : $l < 1$.

We choose a real number r such that $l < r < 1$. (Density property of **R** guarantees such a number)

We write $\varepsilon = r - l$. Then $\varepsilon > 0$. For this ε, there exists $m \in \mathbf{N}$ such that

$$\left|\frac{a_{n+1}}{a_n} - l\right| < \varepsilon \qquad \text{for all } n \geq m$$

That is,

$$l - \varepsilon < \frac{a_{n+1}}{a_n} < l + \varepsilon \text{ for all } n \geq m$$

Then $\frac{a_{n+1}}{a_n} < r$ for all $n \geq m$, as $l + \varepsilon = r$

Taking $n = m, m+1, \ldots, n-1$ respectively in this inequality and multiplying, we get

$$\frac{a_{m+1}}{a_m} \cdot \frac{a_{m+2}}{a_{m+1}} \cdots \frac{a_n}{a_{n-1}} < r.r.\ldots.r \quad \text{(n-m times)}$$

This gives, $a_n < a_m r^{n-m}$.

or, $a_n < cr^n$ for all $n \geq m$, where $c = \frac{a_m}{r^m}$, a positive constant.

The G.P. series Σcr^n in convergent as $r < 1$.

Hence, by the comparison test and Theorem 3.1.9, Σa_n is convergent.

Case-(ii) : $l > 1$

We choose a real number R such that $l > R > 1$.

We write, $\varepsilon = l - R$. For this $\varepsilon > 0$, there exists $m \in \mathbf{N}$ such that

$$\left|\frac{a_{n+1}}{a_n} - l\right| < \varepsilon \qquad \text{for all } n \geq m.$$

That is,

$$l - \varepsilon < \frac{a_{n+1}}{a_n} < l + \varepsilon \text{ for all } n \geq m.$$

Then $R < \dfrac{a_{n+1}}{a_n}$ for all $n \geq m$ as $l - \varepsilon = R$.

Proceeding as in case-(i), we get

$$cR^n < a_n, \qquad \text{for some positive constant c and for all } n \geq m.$$

Since, $R > 1$, Σa_n is divergent, by the comparison test and Theorem 3.1.9.

Case-(iii). $l = 1$

Consider the two series $\Sigma\dfrac{1}{n}$ and $\Sigma\dfrac{1}{n^2}$.

In $\Sigma\dfrac{1}{n}$, we have $\lim\left(\dfrac{a_{n+1}}{a_n}\right) = \lim\left(\dfrac{n}{n+1}\right) = 1$

In $\Sigma\dfrac{1}{n^2}$, we have $\lim\left(\dfrac{a_{n+1}}{a_n}\right) = \lim\left(\dfrac{n}{n+1}\right)^2 = 1$.

But $\Sigma\dfrac{1}{n}$ is divergent and $\Sigma\dfrac{1}{n^2}$ is convergent, as $\Sigma\dfrac{1}{n^p}$ converges for $p > 1$ and diverges for $p \leq 1$.

Hence, no conclusion can be drawn if $l = 1$. Thus the test fails if $l = 1$.

Example 3.3.2. Test for convergence the following series

(a) $\displaystyle\sum\frac{n!}{1.3.5.7...(2n-1)}$ (b) $\displaystyle\sum\frac{1.3.5....(2n-1)}{1.4.7....(3n-2)}$

Solution.

(a) $\displaystyle\lim\left(\frac{a_{n+1}}{a_n}\right) = \lim\left(\frac{(n+1)!}{1.3.5...(2n+1)}.\frac{1.3.5...(2n-1)}{n!}\right)$

$$= \lim\left(\frac{n+1}{2n+1}\right)$$

$$= \frac{1}{2} < 1$$

Hence, by the ratio test, the series converges.

(b) $$\lim\left(\frac{a_{n+1}}{a_n}\right) = \lim\left(\frac{1.3.5...(2n+1)}{1.4.7...(3n+1)} . \frac{1.4.7...(3n-2)}{1.3.5...(2n-1)}\right)$$

$$= \lim\left(\frac{2n+1}{3n+1}\right) = \frac{2}{3} < 1$$

Hence, by the ratio test, the sequence converges.

Example 3.3.3. Test for convergence the series $\sum n^p r^n$ for any real numbers p and r with $0 < r < 1$. Hence, conclude about $\lim\limits_{n\to\infty} \left(n^p r^n\right)$.

Solution. Here, $a_n = n^p r^n$

Hence,

$$\lim\left(\frac{a_{n+1}}{a_n}\right) = \lim\left(\frac{(n+1)^p r^{n+1}}{n^p r^n}\right)$$

$$= \lim\left(\frac{n+1}{n}\right)^p r$$

$$= r < 1$$

Hence, the series is convergent, by the ratio test.

Accordingly, by the n^{th} term test, $\lim(a_n) = 0$.

That is $\lim\limits_{n\to\infty} (n^p r^n) = 0$.

Example 3.3.4. Test the following series for convergence

$$1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + ...$$

for all real x.

Solution. If x = 0, the series converges trivially.

So we assume $x \neq 0$. Then, it is a positive term series with

$$a_n = \frac{x^{2n-2}}{2n-2} \quad \text{for } n \geq 2.$$

and $$a_{n+1} = \frac{x^{2n}}{2n}$$

$$\therefore \quad \lim\left(\frac{a_{n+1}}{a_n}\right) = \lim\left(\frac{2n-2}{2n}x^2\right) = x^2$$

By the ratio test, the series converges if $x^2 < 1$ and diverges if $x^2 > 1$.

This means the series is convergent for $|x| < 1$ and divergent for $|x| > 1$.

For $|x| = 1$, the given series becomes

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots \quad \text{where } a_n = \frac{1}{2n-2} \quad \text{for } n \geq 2.$$

We take $b_n = \frac{1}{n}$. Then

$$\lim\left(\frac{a_n}{b_n}\right) = \lim\left(\frac{n}{2n-2}\right) = \frac{1}{2}$$

Since $\lim\left(\frac{a_n}{b_n}\right)$ is non-zero and finite, Σa_n and Σb_n converge or diverge together. But the series $\sum \frac{1}{n}$ is divergent as $\sum \frac{1}{n^p}$ diverges for $p \leq 1$. Therefore Σa_n diverges.

Thus, the given series converges for $|x| < 1$ and diverges for $|x| \geq 1$.

Theorem (Cauchy's n^{th} root test) 3.3.5. Let Σa_n be a positive term series and let

$$\lim (a_n)^{1/n} = l.$$

Then,

(i) If $l < 1$, the series Σa_n converges;

(ii) If $l > 1$, the series Σa_n diverges ;

(iii) If $l = 1$, the test fails.

Proof. Case-(i) : $l < 1$.

We choose a real number r such that $l < r < 1$. (Density property of **R** guarantees such r).

We write $\varepsilon = r - l$. Then $\varepsilon > 0$.

Since, $\lim (a_n)^{1/n} = l$, there exists $m \in \mathbf{N}$ such that

$$\left| a_n^{1/n} - l \right| < \varepsilon \qquad \text{for all } n \geq m$$

or $\quad l - \varepsilon < a_n^{1/n} < l + \varepsilon \quad$ for all $n \geq m$.

Then, $a_n^{1/n} < r \qquad$ for all $n \geq m$ as $l + \varepsilon = r$.

or, $\quad a_n < r^n \qquad$ for all $n \geq m$.

Now, the G.P. series Σr^n converges as $r < 1$.

Hence, by the comparison test and Theorem 3.1.9, Σa_n converges.

Case-(ii) : $l > 1$

We choose a real number R such that $l > R > 1$ and we write $\varepsilon = l - R$. Then $\varepsilon > 0$.

Proceeding as in Case-(i), and replacing $l - \varepsilon$ by R, we find that there exists $m_0 \in \mathbf{N}$ such that

$$R^n < a_n \quad \text{for all } n \geq m_0.$$

Now, ΣR^n is divergent as $R > 1$. Hence, by the comparison test and Theorem 3.1.9, Σa_n is divergent.

Case-(iii) : $l = 1$

Consider the series $\Sigma \frac{1}{n}$ and $\Sigma \frac{1}{n^2}$. We have

$$\lim \left(\frac{1}{n} \right)^{1/n} = \lim \frac{1}{(n)^{1/n}} = 1$$

and $$\lim \left(\frac{1}{n^2} \right)^{1/n} = \lim \frac{1}{(n^{1/n})^2} = 1$$

However, as discussed earlier, $\Sigma \frac{1}{n}$ is divergent and $\Sigma \frac{1}{n^2}$ is convergent. Hence, no conclusion can be drawn when $l = 1$.

Example 3.3.6. Test for the convergence the following series

(i) $\sum (n^{1/n} - 1)^n$ (ii) $\sum \frac{n^{n^2}}{(n+1)^{n^2}}$

Solution. (i) $a_n = (n^{1/n} - 1)^n$

$$\lim (a^n)^{1/n} = \lim (n^{1/n} - 1)$$

$$= \lim (n^{1/n}) - 1 = 1 - 1 = 0 < 1$$

Hence, by the n^{th} root test, the series is convergent.

(ii) Here $a_n = \frac{n^{n^2}}{(n+1)^{n^2}}$

$$\lim (a_n)^{1/n} = \lim \frac{n^n}{(n+1)^n}$$

$$= \lim \frac{1}{\left(1 + \frac{1}{n}\right)^n} = \frac{1}{e} < 1$$

Hence, by the n^{th} root test, the given series is convergent.

ILLUSTRATIVE EXAMPLES

1. Determine whether the following series converge or diverge

(a) $\sum \frac{n+1}{2^n}$ (b) $\sum \frac{(n!)^2}{(2n)!}$ (c) $\sum \frac{n10^{n+1}}{\pi^{2n}}$

(d) $\sum \left(\frac{2n^2}{3n^2+1}\right)^n$ (e) $\sum \frac{n!}{n^3+4}$ (f) $\sum \frac{n!}{n^n}$

(g) $\sum \frac{n^3}{n!}$ (h) $\sum \frac{n^2}{2^n}$ (i) $\sum ne^{-n^2}$

(j) $\sum \sin^n\left(\frac{1}{n}\right)$ (k) $\sum \frac{n!}{e^n}$ (l) $\sum \frac{1}{(\ln(n+1))^n}$, $n \geq 2$

Solution. (a) $a_n = \frac{n+1}{2^n}$

$$\frac{a_{n+1}}{a_n} = \frac{n+2}{2^{n+1}} \cdot \frac{2^n}{n+1} = \frac{1}{2}\left(\frac{1+\frac{2}{n}}{1+\frac{1}{n}}\right)$$

$$\lim\left(\frac{a_{n+1}}{a_n}\right) = \frac{1}{2} < 1$$

Therefore, by the ratio test, the series is convergent.

(b) $$a_n = \frac{(n!)^2}{(2n)!}$$

$$\frac{a_{n+1}}{a_n} = \left(\frac{(n+1)!}{n!}\right)^2 \frac{(2n)!}{(2n+2)!}$$

$$= \frac{(n+1)^2}{(2n+1)(2n+2)}$$

$$= \frac{n+1}{2(2n+1)}$$

$$\lim\left(\frac{a_{n+1}}{a_n}\right) = \lim\left(\frac{n+1}{2(2n+1)}\right) = \frac{1}{4} < 1$$

Therefore, by the ratio test, the series is convergent.

(c) **Hint.** $a_n = \frac{n10^{n+1}}{\pi^{2n}}$, apply the ratio test or the root test.

(d) $$a_n = \left(\frac{2n^2}{3n^2+1}\right)^n = \left(\frac{2}{3+\frac{1}{n^2}}\right)^n$$

$$(a_n)^{1/n} = \frac{2}{3+\frac{1}{n^2}}$$

$$\lim (a_n)^{1/n} = \frac{2}{3} < 1.$$

Hence, by the root test, the series is convergent.

(e) $$a_n = \frac{n!}{n^3+4}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^3+4} \cdot \frac{n^3+4}{n!}$$

$$= (n+1)\frac{1+\frac{4}{n^3}}{\left(1+\frac{1}{n}\right)^3+\frac{4}{n^3}}$$

$$\lim\left(\frac{a_{n+1}}{a_n}\right) = \infty$$

Hence, the series is divergent, by the ratio test.

(f) $a_n = \frac{n!}{n^n}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$$

$$= \frac{1}{(n+1)^n} . n^n$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$\lim\left(\frac{a_{n+1}}{a_n}\right) = \frac{1}{e} < 1.$$

Hence, the series is convergent.

Apply ratio test or root test in the remaining parts.

2.	Let Σa_n be a positive term series. If $\lim\left(\frac{a_{n+1}}{a_n}\right) = l$, then show that $\lim (a_n)^{1/n} = l$.

Solution. Let $\varepsilon > 0$ be given. Then there exists $m \in \mathbf{N}$ such that

$$\left|\frac{a_{n+1}}{a_n} - l\right| < \varepsilon \qquad \text{for all } n \geq m$$

or $\quad l - \varepsilon < \frac{a_{n+1}}{a_n} < l + \varepsilon \qquad$ for all $n \geq m$.

We write, $l + \varepsilon = M$, $l - \varepsilon = N$. Then

$$N < \frac{a_{n+1}}{a_n} < M \qquad \text{for} \qquad \text{all } n \geq m \qquad \text{(i)}$$

Taking n = m, m + 1, …, n - 1 in (i) and then multiplying, we get

$$N.N....N < \frac{a_{m+1}}{a_m} . \frac{a_{m+2}}{a_{m+1}} ... \frac{a_n}{a_{n-1}} < M.M....M \quad \text{(n - m times)}$$

$$\Rightarrow \qquad N^{n-m} < \frac{a_n}{a_m} < M^{n-m}$$

$$\Rightarrow \qquad \frac{a_m}{N^m}.N^n < a_n < \frac{a^m}{M^m} M^n$$

$$\Rightarrow \qquad cN^n < a_n < dM^n, \qquad \text{where } c = \frac{a_m}{N^m},\ d = \frac{a_m}{M^m}$$

If N > 0, we take the n^{th} root of each part of the above inequality to get

$$c^{1/n} N < a_n^{1/n} < d^{1/n} M$$

Using property of limit superior and limit inferior (See Exercises 2.8), we get

$$\liminf (c^{1/n} N) \leq \liminf (a_n^{1/n}) \leq \limsup (a_n^{1/n}) \leq \limsup (d^{1/n} M)$$

Since $c^{1/n} \to 1$ and $d^{1/n} \to 1$, we get

$$\liminf (c^{1/n} N) = N, \ \limsup (d^{1/n} M) = M.$$

Hence, $N \leq \liminf (a_n^{1/n}) \leq \limsup (a_n^{1/n}) \leq M$.

If $N \leq 1$, then clearly $\liminf (a_n^{1/n}) \geq N$ as $a_n > 0$ for all n.

Thus, we have

$$l - \varepsilon \leq \liminf (a_n^{1/n}) \leq \limsup (a_n^{1/n}) \leq l + \varepsilon \qquad \text{[using (i)]}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\liminf (a_n^{1/n}) = \limsup (a_n^{1/n}) = l$$

Hence, $\lim (a_n^{1/n}) = l$.

Remark. The above result suggests that the root test is at least as strong as the ratio test in the sense that if the ratio test proves the convergence or divergence of a positive term series, the root test will also do it. The following example shows that the root test is actually stronger than the ratio test.

3. Show that the series $\Sigma 2^{(-1)^n - n}$ is convergent by using the root test, although the ratio test fails for this series.

Solution. Here $a_n = 2^{(-1)^n - n}$

$$= \begin{cases} 2^{-1-n}, & \text{if } n \text{ is odd} \\ 2^{1-n}, & \text{if } n \text{ is even} \end{cases}$$

Hence, $(a_n)^{1/n} = \begin{cases} 2^{(-1/n)-1}, & \text{if } n \text{ is odd} \\ 2^{(1/n)-1}, & \text{if } n \text{ is even} \end{cases}$

Therefore, $\lim (a_n)^{1/n} = \frac{1}{2}$ as $\lim (2^{1/n}) = \lim (2^{-1/n}) = 1$.

Thus, the series is convergent by the root test.

Now, $\frac{a_{n+1}}{a_n} = \begin{cases} \dfrac{2^{1-(n+1)}}{2^{-1-n}}, & n \text{ is odd} \\ \dfrac{2^{-1-(n+1)}}{2^{1-n}}, & n \text{ is even} \end{cases}$

$$= \begin{cases} 2, & n \text{ is odd} \\ \frac{1}{8}, & n \text{ is even} \end{cases}$$

Hence, $\lim \left(\frac{a_{n+1}}{a_n} \right)$ does not exist. Therefore, the ratio test cannot be applied for this series.

4. Discuss the convergence of the series

$$\sum \frac{1}{p^n - q^n}, \qquad 0 < q < p.$$

Solution. Here $a_n = \frac{1}{p^n - q^n}$

$$\frac{a_n}{a_{n+1}} = \frac{p^{n+1} - q^{n+1}}{p^n - q^n}$$

$$= \frac{(p-q)(p^n + p^{n-1}q + \ldots + q^n)}{p^n - q^n}$$

$$= \frac{(p-q)p^n\left(1+\frac{q}{p}+\left(\frac{q}{p}\right)^2+\ldots+\left(\frac{q}{p}\right)^n\right)}{p^n\left(1-\left(\frac{q}{p}\right)^n\right)}$$

$$= \frac{(p-q)}{1-\left(\frac{q}{p}\right)^n}\left(\frac{1-\left(\frac{q}{p}\right)^{n+1}}{1-\frac{q}{p}}\right)$$

As $0 < q < p$, we have $\frac{q}{p} < 1$, so that $\lim\left(\frac{q}{p}\right)^n = 0$.

Hence, we get,

$$\lim\left(\frac{a_n}{a_{n+1}}\right) = \frac{p-q}{1-\frac{q}{p}} = p \quad \text{or} \quad \lim\left(\frac{a_{n+1}}{a_n}\right) = \frac{1}{p}$$

Therefore, by the ratio test, the series converges if $p > 1$ and diverges if $p < 1$.

The test fails for $p = 1$.

If $p = 1$, we have

$$a_n = \frac{1}{1-q^n}$$

$$\lim(a_n) = 1 \quad \text{as } 0 < q < p = 1.$$

Therefore, the series diverges by the n^{th} term test.

Hence, the series diverges for $p \le 1$ and converges for $p > 1$.

EXERCISES 3.3

1. Test the following series for convergence

(a) $\sum \frac{5^n}{n^2+5}$ (b) $\sum \frac{2^{n-1}}{3^n+1}$ (c) $\sum \frac{n!}{n^n}$

(d) $\sum \frac{n^3}{(n+1)!}$ (e) $\sum \frac{3^n}{n^2+3}$ (f) $\sum 2^{-n-(-1)^n}$

(g) $\sum \frac{1}{(\ln n)^n}$ (h) $\sum \frac{1}{n^n}$ (i) $\sum \left(1-\frac{1}{n}\right)^n$

(j) $\sum e^{-n^2}$ (k) $\sum \frac{(n+1)^n}{n^{2n}}$ (l) $\sum \frac{n!x^n}{n^n}$, $x \neq e$

(m) $\sum \left(\sin^{-1}\left(\frac{1}{n}\right)\right)^n$ (n) $\sum \frac{n^3+p}{3^n+q}$ (o) $\sum \frac{(\cot^{-1} n)^n}{2^n}$

(p) $\sum \frac{1}{\sqrt{n!}} e^n$

2. Discuss the convergence of the series

$$1+\frac{\alpha+1}{\beta+1}+\frac{(\alpha+1)(2\alpha+1)}{(\beta+1)(2\beta+1)}+\frac{(\alpha+1)(2\alpha+1)(3\alpha+1)}{(\beta+1)(2\beta+1)(3\beta+1)}+\ldots$$

3. Construct a series Σa_n such that $\lim\left(\frac{a_{n+1}}{a_n}\right)$ does not exist, but

(i) Σa_n is convergent;

(ii) Σa_n is divergent.

4. Construct a series Σa_n, $a_n > 0$, such that $\lim (a_n)^{1/n}$ does not exist, but

(a) Σa_n converges;

(b) Σa_n diverges.

5. Test for convergence the following series

(a) $x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\ldots \quad (x \geq 0)$

(b) $\sum \frac{(a+x)^n}{n!} \quad (a, x > 0)$

(c) $\sum \frac{x^n}{n!}$ (d) $1+\frac{x^2}{2}+\frac{x^4}{4}+\frac{x^6}{6}+\ldots.$

(e) $\frac{x}{1.3}+\frac{x^2}{2.4}+\frac{x^3}{3.5}+\frac{x^4}{4.6}+\ldots \quad (x > 0)$

(f) $\sum \frac{1.3.5...(2n+1)}{2.5.8...(3n+2)}$ (g) $\sum \frac{1.3.5...(2n-1)}{n!3^n}$

(h) $\sum \frac{\sqrt{n-1}}{\sqrt{n^3+1}} x^n$, $x > 0$ (i) $\sum \frac{x^{n+1}}{(n+1)\sqrt{n}}$

(j) $\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + ...$

(k) $\sum \frac{(n - \ln n)^n}{2^n n^n}$ (l) $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \frac{4^4}{5^5} + ...$

(m) $\sum \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$ (n) $\frac{1^2.2^2}{1!} + \frac{2^2.3^2}{2!} + \frac{3^2.4^2}{3!} + ...$

(o) $1 + \frac{2}{5}x + \frac{6}{9}x^2 + \frac{14}{17}x^3 + \frac{30}{33}x^4 + ...$ $(x > 0)$.

3.4 INTEGRAL TEST

In this test, integration is used to determine convergence of series.

Theorem (Integral Test) 3.4.1. Let Σa_n be a series of positive terms and suppose that there exists a positive, decreasing, integrable function f defined over $[1, \infty)$ such that $f(n) = a_n$ for every $n \in \mathbf{N}$, then the series Σa_n converges if and only if the sequence (I_n) converges, where

$$I_n = \int_1^n f(x)dx$$

(In other words, Σa_n converges if and only if $\int_1^\infty f(x)dx$ converges).

Proof. Since f(x) is a decreasing function for $x \geq 1$, we have, for $k \in \mathbf{N}$

$$f(k) \geq f(x) \geq f(k+1), \text{ whenever } k \leq x \leq k + 1$$

$$\Rightarrow \quad a_k \geq \int_k^{k+1} f(x)dx \geq a_{k+1} \qquad [\because f(x) \leq g(x) \text{ on } [a,b] \Rightarrow \int_a^b f(x)dx \leq \int_a^b f(x)dx]$$

Taking k = 1,2,…,n-1 in succession and adding, we get

$$\sum_{k=1}^{n-1} a_k \geq \int_1^n f(x)dx \geq \sum_{k=1}^{n} a_k - a_1$$

or $\quad S_{n-1} \geq I_n \geq S_n - a_1, \quad$ where $S_n = a_1 + a_2 + \ldots + a_n \qquad$ (i)

Since $I_{n+1} - I_n = \int_n^{n+1} f(x)dx \geq 0$,

(I_n) is an increasing sequence.

Now, suppose that (I_n) converges and $\lim(I_n) = I$.

Then, $I \geq I_n$ as $I = \sup\{I_n : n \in \mathbf{N}\}$, by the Monotone Convergence Theorem.

Hence, from (i), we get

$$S_n \leq I_n + a_1 \leq I + a_1$$

Thus, (S_n) is bounded above.

As (S_n) is an increasing sequence, it is convergent by the Monotone Convergent Theorem.

Hence, Σa_n is convergent.

Conversely, suppose Σa_n converges. That is, (S_n) is convergent.

Let $\quad \lim(S_n) = S$. Then using (i), we get

$$I_n \leq S_{n-1} \leq S \qquad \text{for all } n \geq 2.$$

Thus, (I_n) is bounded above.

Hence, it is convergent, by the Monotone Convergent Theorem.

Remark. The domain of f in the above theorem can be $[m, \infty)$ for any positive integer m. The result will hold for $\sum_{n=m}^{\infty} a_n$ and hence for $\sum_{n=1}^{\infty} a_n$, in view of Theorem 3.1.9.

The integral test is particularly useful in determining the convergence of the so called p-series $\sum \frac{1}{n^p}$ for different values of p.

Example 3.4.2. The series $\sum \frac{1}{n^p}$ is convergent for $p > 1$ and diverges for $p \leq 1$.

Solution. If $p \le 0$, then the sequence $\left(\frac{1}{n^p}\right) = \left(n^{-p}\right)$ diverges to ∞. Hence, by the n^{th} term test, $\sum \frac{1}{n^p}$ diverges.

For $p \ge 0$, the function $f(x) = \frac{1}{x^p}$ is positive and decreasing in $[1, \infty)$. Now, we have

$$I_n = \int_1^n f(x)dx$$

$$= \int_1^n \frac{1}{x^p} dx$$

$$= \begin{cases} \ln x\Big]_1^n & \text{if } p = 1 \\ \left.\dfrac{x^{-p+1}}{-p+1}\right]_1^n & \text{if } p \ne 1 \end{cases}$$

$$= \begin{cases} \ln n & \text{if } p = 1 \\ \dfrac{1}{1-p}\left[n^{1-p} - 1\right], & \text{if } p \ne 1 \end{cases}$$

$$\therefore \qquad \lim (I_n) = \begin{cases} \infty & \text{if } p \le 1 \\ \dfrac{1}{p-1} & \text{if } p > 1 \end{cases}$$

Hence, by the integral test, the p-series converges if $p > 1$ and diverges if $p \le 1$.

Example 3.4.3. Discuss the convergence of the series

$$\frac{1}{2\ln 2} + \frac{1}{3\ln 3} + \frac{1}{4\ln 4} + \frac{1}{5\ln 5} + \dots$$

Solution. We write, $a_n = \frac{1}{n \ln n}$, $n \ge 2$ and take $a_1 = 0$. If we take

$$f(x) = \frac{1}{x \ln x} \qquad \text{for } x \ge 2$$

then f is positive, decreasing integrable function with $f(n) = a_n$.

Now, $I_n = \int_2^n f(x)dx = \int_2^n \frac{1}{x \ln x} dx$

$$= \ln \ln x\Big]_2^n$$

$$= \ln \ln n - \ln \ln 2$$

$\therefore \qquad \lim (I_n) = \infty$.

That is, the sequence (I_n) is divergent.

Hence, by the integral test, the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ is divergent.

ILLUSTRATIVE EXAMPLES

1. Discuss the convergence of the following series

(a) $\sum_{n=1}^{\infty} ne^{-n^2}$ (b) $\sum_{n=1}^{\infty} ne^{-n}$

Solution. (a) $a_n = ne^{-n^2}$.

Consider the function $f(x) = xe^{-x^2}$ for $x \geq 1$

Then, $f'(x) = (1-2x^2)e^{-x^2} < 0$ for $x \geq 1$.

Thus, f is monotonic decreasing, positive, integrable function with $f(n) = a_n$ for all $n \in \mathbf{N}$.

$$I_n = \int_1^n xe^{-x^2} dx = \frac{1}{2}\left[e^{-1} - e^{-n^2}\right]$$

$$\lim(I_n) = \frac{1}{2e}$$

Hence, by the integral test, the series Σne^{-n^2} is convergent.

(b) $a_n = ne^{-n}$

Consider the function $f(x) = xe^{-x}$ and proceed as above.

2. Discuss the convergence of the series

$$\frac{1}{2(\ln 2)^p} + \frac{1}{3(\ln 3)^p} + \frac{1}{4(\ln 4)^p} + \dots, \ p > 0$$

Solution. Let $f(x) = \frac{1}{x(\ln x)^p}$, $p > 0$

Since $p > 0$, therefore for $x \geq 2$, f is a non-negative, monotonically decreasing, integrable function such that

$$f(n) = \frac{1}{n(\ln n)^p}, \quad \text{for all } n \geq 2.$$

Therefore, by the integral test

$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ and (I_n) where $I_n = \int_2^n f(x)dx$ converge or diverge together.

Now, $I_n = \int_2^n f(x)dx$

$$= \int_2^n \frac{dx}{x(\ln x)^p}$$

$$= \begin{cases} \ln(\ln n) - \ln(\ln 2), & \text{if } p = 1 \\ \dfrac{(\ln n)^{1-p} - (\ln 2)^{1-p}}{1-p}, & \text{if } p \neq 1 \end{cases}$$

$$\lim_{n\to\infty} (I_n) = \begin{cases} \infty, & \text{if } p \leq 1 \\ \dfrac{(\ln 2)^{1-p}}{p-1}, & \text{if } p > 1 \end{cases}$$

Thus, (I_n) converges for $p > 1$ and diverges for $0 < p \leq 1$.

Therefore, the given series converges for $p > 1$ and diverges for $0 < p \leq 1$.

3. Show that $\sum \frac{\ln n}{n^p}$ is convergent if and only if $p > 1$.

Solution. If $p \leq 0$, then $a_n \to \infty$.

Therefore, we take $p > 0$.

Let $f(x) = \frac{\ln x}{x^p}$.

Then, $f'(x) = \frac{x^{p-1} - \ln x(px^{p-1})}{x^{2p}}$

$$= \frac{x^{p-1}(1 - p\ln x)}{x^{2p}}$$

$\Rightarrow$ $f'(x) \leq 0$ if $p\ln x \geq 1$

that is, if $x \geq e^{1/p}$.

We can choose a positive integer m such that $m > e^{1/p}$.

Then, f is a decreasing function for all $x \geq m$.

Now, $$\int \frac{\ln x}{x^p} dx = \begin{cases} \ln x\left(\dfrac{x^{-p+1}}{-p+1}\right) - \displaystyle\int \frac{x^{-p}}{1-p} dx, & p \neq 1 \\ \dfrac{1}{2}(\ln x)^2, & p = 1 \end{cases}$$

$$= \begin{cases} \frac{1}{1-p} \ln x (x^{1-p}) - \frac{1}{(1-p)^2} x^{1-p}, & p \neq 1 \\ \frac{1}{2} (\ln x)^2 & p = 1 \end{cases}$$

$$= \begin{cases} x^{1-p} \left(\frac{\ln x}{1-p} - \frac{1}{(1-p)^2} \right), & p \neq 1 \\ \frac{1}{2} (\ln x), & p = 1 \end{cases}$$

These expressions tend to ∞ as x tends to ∞ if $p \leq 1$.

If $p > 1$, then as x tends to ∞, the first expression tends to 0.

Therefore, $\int_m^n \frac{\ln x}{x^p} dx$ converges if and only if $p > 1$.

Hence, by the integral test, the series $\sum \frac{\ln n}{n^p}$ converges if and only if $p > 1$.

EXERCISES 3.4

1. Discuss the convergence of the following series

   (a) $\sum_{n=3}^{\infty} \frac{\ln n}{n}$ (b) $\sum_{n=3}^{\infty} \frac{1}{n \ln n}$

   (c) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{3/2}}$ (d) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1/2}}$

   (e) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

2. Discuss the convergence of the following series

   (a) $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ (b) $\sum_{n=1}^{\infty} n e^{-2n}$

3. Show that $\sum_{n=1}^{\infty} \frac{1}{(n+1)^p}$ is convergent, if $p > 1$ and divergent if $p \leq 1$.

4. Determine the convergence of the series

   $$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{2p}}, \quad p > 0.$$

5. Using Cauchy's integral test, discuss the convergence of the series whose n^{th} term is

$$\frac{1}{n(\ln n)(\ln \ln n)^p}, \; n \geq 3, \; p > 0.$$