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*B SC(H) MATHEMATICS 1<sup>ST</sup> YR(SEC-A & SEC-B)*  
*PAPER:REAL ANALYSIS*

*DR R D SARMA*

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### UNIT-3

## INFINITE SERIES

We are familiar with the terms 'geometric series' or G.P. series, 'Arithmetic series' or A.P. series etc. These series are usually expressed as  $\sum_{n=1}^{\infty} a_n$  or  $a_1 + a_2 + a_3 + \dots$  and so

on. They are examples of infinite series. However to denote an infinite series by  $\sum_{n=1}^{\infty} a_n$

or  $a_1 + a_2 + a_3 + \dots + a_n + \dots$  is rather customary and is intuitive in nature. It is because only finite sums of the type  $a_1 + a_2 + \dots + a_n$  is well-defined under the binary operation of addition of real numbers. Nevertheless, our knowledge of convergence of sequences envisages a logical way of defining an infinite series and its convergence.

### 3.1 CONVERGENCE OF INFINITE SERIES

**Definition 3.1.1.** An *infinite series*, denoted by  $\sum_{n=1}^{\infty} a_n$ , is an ordered pair  $((a_n), (S_n))$

where  $(a_n)$  is a sequence of real numbers and  $S_n = a_1 + a_2 + \dots + a_n$ , for each  $n \in \mathbf{N}$ . The number  $a_n$  is called the  $n^{\text{th}}$  term of the series and  $S_n$  is called the  $n^{\text{th}}$  partial sum of the series.

In addition to  $\sum_{n=1}^{\infty} a_n$ , we sometimes write  $\sum a_n$ ,  $a_1 + a_2 + \dots + a_n + \dots$  or simply  $a_1 + a_2 + \dots$  to denote the series  $((a_n), (S_n))$ .

The definition of convergence of a series  $\sum_{n=1}^{\infty} a_n$  is based on the notion of convergence of the sequence  $(S_n)$  of its partial sums.

**Definition 3.1.2.** Let  $\sum_{n=1}^{\infty} a_n$  be an infinite series of real numbers with its  $n^{\text{th}}$  partial sum  $S_n = a_1 + a_2 + \dots + a_n$ . The series is said to be *convergent* if the sequence  $(S_n)$  converges. If  $(S_n)$  diverges, we say that the series  $\sum_{n=1}^{\infty} a_n$  is divergent.

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If  $(S_n)$  converges to  $S$ , we often write  $\sum_{n=1}^{\infty} a_n = S$  and define  $S$  to be the *sum of the series*. Thus, we use the notation  $\sum_{n=1}^{\infty} a_n$ , not only to denote the series  $((a_n), (S_n))$ , but also to denote its sum, in case the series is convergent. The reader must be aware of the use (rather abuse !) of the notation  $\sum_{n=1}^{\infty} a_n$  or, to be simply written, of  $\Sigma a_n$ , from its context.

**Example 3.1.3(a).** The series  $\sum_{n=1}^{\infty} 1$ , that is, the series  $1 + 1 + \dots$  is divergent as its sequence of partial sums  $(S_n)$  is divergent, where  $S_n = n$  in this case.

(b) Consider the series

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{n(n+1)} + \dots$$

Here,  $a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

and  $S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$

Since,  $\lim (S_n) = \lim \left(1 - \frac{1}{n+1}\right) = 1$ , the sequence  $(S_n)$  of partial sums is convergent.

Therefore, the series is convergent.

(Series of the above type where all but a few terms cancel, are called *telescoping series*)

(c) The series  $\Sigma(-1)^{n+1}$  has the sequence of partial sums  $(S_n)$ , given by

$$\begin{aligned} S_n &= 0 && \text{where } n \text{ is even} \\ &= 1 && \text{when } n \text{ is odd} \end{aligned}$$

That is,  $(S_n) = (1, 0, 1, 0, \dots)$ . Clearly  $(S_n)$  is divergent.

Hence, the series is divergent.

**Example 3.1.4.** If  $a \neq 0$ , then the geometric series  $\sum_{k=1}^{\infty} ar^{k-1}$  converges if and only if

$|r| < 1$  and when converges, the sum is  $\frac{a}{1-r}$ .

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**Solution.** The  $n^{\text{th}}$  partial sum,

$$\begin{aligned} S_n &= a(1 + r + \dots + r^{n-1}) \\ &= a \frac{(1-r^n)}{1-r} \quad (\text{for } r \neq 1) \end{aligned}$$

Hence  $(S_n)$  converges if and only if  $(r^n)$  converges, when  $r \neq 1$ .

From Example 2.3.6, it follows that  $\lim(r^n) = 0$  if  $|r| < 1$ .

For  $|r| > 1$ ,  $(r^n)$  diverges.

Also, for  $r = 1$ ,  $S_n = na$  and hence  $(S_n)$  diverges.

Thus, the series converges if and only if  $|r| < 1$ . In that case,

$$\lim(S_n) = \lim a \left( \frac{1-r^n}{1-r} \right) = \frac{a}{1-r}.$$

**Example 3.1.5.** Find out for what values of  $x \in \mathbf{R}$ , the following series converges

(a)  $\sum \sin^n x$       (b)  $\sum e^{nx}$

**Solution.** (a) It is a geometric series with  $r = \sin x$ .

Hence, the series converges where  $|\sin x| < 1$ .

Since,  $\sin x = \pm 1$  for  $x = \pm(2n-1)\frac{\pi}{2}$ ,  $n \in \mathbf{N}$ , therefore the given series converges in

whole of  $\mathbf{R}$  except at the points  $x = \pm(2n-1)\frac{\pi}{2}$ ,  $n \in \mathbf{N}$ .

(b)  $e^{nx} = (e^x)^n$ . Here the given series is a geometric series with  $r = e^x$ . Now  $f(x) = e^x$  is an increasing function with  $0 < e^x < 1$  for  $x \in (-\infty, 0)$ .

Hence, the series  $\sum e^{nx}$  converges for all  $x \in (-\infty, 0)$ .

The following result provides a necessary condition for the convergence of a series.

**Theorem ( $n^{\text{th}}$  term test) 3.1.6.** If a series  $\sum a_n$  converges, then the sequence  $(a_n)$  converges to 0.

**Proof.** Let  $S_n = a_1 + a_2 + \dots + a_n$ . Then

$$a_n = S_n - S_{n-1} \quad (\text{i})$$

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Since,  $\Sigma a_n$  is convergent,  $\lim (S_n)$  exists. Let  $\lim (S_n) = S$ .

Now, taking limit on both sides of (i), we get ,

$$\begin{aligned}\lim (a_n) &= \lim (S_n - S_{n-1}) \\ &= \lim(S_n) - \lim(S_{n-1}) \\ &= S - S = 0\end{aligned}$$

Hence,  $(a_n)$  converges to 0 whenever  $\Sigma a_n$  converges.

**Example 3.1.7.** Is the series  $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$  convergent ?

**Solution.** Here  $a_n = \cos\left(\frac{1}{n}\right)$ .

Therefore,  $\lim (a_n) = \lim\left(\cos\left(\frac{1}{n}\right)\right) = 1$ .

Hence, by the  $n^{\text{th}}$  term test, the given series is divergent.

**Example 3.1.8.** For what values of  $x \in \mathbf{R}$ , is the series  $\sum_{n=1}^{\infty} \sin nx$  convergent ?

**Solution.** We know that  $\sin nx = 0$  only when  $x$  is a member of  $\{0, \pm \pi, \pm 2\pi, \dots\}$ .

Thus, by the  $n^{\text{th}}$  term test,  $\Sigma \sin nx$  diverges if  $x \notin \{0, \pm \pi, \pm 2\pi, \dots\}$ .

When  $\sin nx = 0$ , the sequence of partial sum becomes a constant sequence consisting of o's and hence is convergent.

Thus, the series  $\Sigma \sin nx$  converges if and only if  $x$  is an integral multiple of  $\pi$ .

The following results will be frequently used in proving further results about series. These results are quite general in nature and may be termed as '*algebraic properties*' of series.

**Theorem 3.1.9.** (a) Any finite number of terms can be added to or removed from a series without affecting its convergence or divergence.

(b) If  $c \neq 0$ , then  $\Sigma ca_n$  converges if and only if  $\Sigma a_n$  converges. When convergent, we have

$$\Sigma ca_n = c\Sigma a_n$$

(c) If  $\Sigma a_n$  and  $\Sigma b_n$  converges with sums  $A$  and  $B$  respectively, then the series  $\Sigma(a_n + b_n)$  converges with sum  $A + B$ .

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That is,

$$\Sigma(a_n + b_n) = \Sigma a_n + \Sigma b_n$$

**Proof.** We prove only part (a); the rest are left as exercises for the students.

Suppose that  $a_{n_1}, a_{n_2}, \dots, a_{n_m}$  have been removed from the series  $\Sigma a_n$ .

Let  $\alpha = a_{n_1} + a_{n_2} + \dots + a_{n_m}$ . Further we suppose, for simplicity, that  $n_1 < n_2 < \dots < n_m$ .

Now, if  $S_n = \sum_{k=1}^n a_k$  and  $T_n$  is the  $n^{\text{th}}$  partial sum of the series formed from  $\Sigma a_n$  after removal of those  $m$  terms, then

$$T_n = S_{n+m} - \alpha$$

provided that  $n + m > n_m$ . Hence,  $\lim(T_n)$  exists if and only if  $\lim(S_{n+m})$  exists.

Now,  $(S_{n+m})$ , being the  $m$ -tail of the sequence  $(S_n)$ , converges if and only if  $(S_n)$  converges. Consequently  $(T_n)$  converges if and only if  $(S_n)$  converges.

The case when the terms are added can be proved in a similar manner.

**Example 3.1.10.** Comment on the convergence of the following series

(a)  $1 + 2 + 3 + 4 + \dots + 100 + \frac{1}{2^{101}} + \frac{1}{2^{102}} + \frac{1}{2^{103}} + \frac{1}{2^{104}} + \dots + \frac{1}{2^n} + \dots$

(b)  $\sum_{n=1}^{\infty} \frac{5}{3^{n-1}}$

(c)  $\sum_{n=1}^{\infty} \left( \frac{3}{5^{n-1}} + \frac{2}{7^{n-1}} \right)$

**Solution.** (a) This series is obtained from the series

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n} + \dots$$

by replacing first 100 terms by the term 1, 2, 3, ..., 100 respectively.

Now,  $\sum_{n=1}^{\infty} \frac{1}{2^n}$  is a geometric series with  $r = \frac{1}{2}$ . Hence, it is convergent. Removal or

addition of any finite number of terms do not affect the convergence of a series.

Hence, the given series is convergent.

(b) The given series is

$$\sum_{n=1}^{\infty} \frac{5}{3^{n-1}} = 5 \sum_{n=1}^{\infty} \frac{1}{3^{n-1}}.$$

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It is obtained by multiplying each term of the series  $\sum \frac{1}{3^{n-1}}$  by 5.

Since,  $\sum \frac{1}{3^{n-1}}$  is convergent, the given series is convergent.

Also, its sum is  $5 \cdot \frac{1}{1 - \frac{1}{3}} = \frac{15}{2}$ .

(c) The series  $\sum \frac{1}{5^{n-1}}$  and  $\sum \frac{1}{7^{n-1}}$  are convergent.

Therefore  $3\sum \frac{1}{5^{n-1}}$  and  $2\sum \frac{1}{7^{n-1}}$  are convergent.

Since, sum of two convergent series is convergent, the given series is convergent.

Also, its sum =  $3 \cdot \frac{1}{1 - \frac{1}{5}} + 2 \cdot \frac{1}{1 - \frac{1}{7}} = \frac{73}{12}$

Since the convergence of a series is defined in terms of the convergence of its sequence of partial sums, Cauchy's Criterion for convergence of sequence can be adopted for series as well. Here is the result.

**Theorem (Cauchy's Convergence Criterion for Series) 3.1.11.** A series  $\sum a_n$  is convergent if and only if for each  $\varepsilon > 0$ , there exists  $p \in \mathbf{N}$  such that

$$|a_{m+1} + a_{m+2} + \dots + a_n| < \varepsilon \text{ for all } n \geq m \geq p.$$

**Proof.** By the Cauchy Criterion of convergence of sequences, the sequence  $(S_n)$  of partial sums of  $\sum a_n$  converges if and only if for each  $\varepsilon > 0$ , there exist  $p \in \mathbf{N}$  such that

$$|S_n - S_m| < \varepsilon \quad \text{for all } n \geq m \geq p$$

or  $|a_{m+1} + a_{m+2} + \dots + a_n| < \varepsilon \quad \text{for all } n \geq m \geq p$

**Example 3.1.12.** Show that the series  $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \dots$  is divergent.

**Solution.** Suppose the given series is convergent. Then it must satisfy Cauchy's Criterion. Accordingly, for  $\varepsilon = \frac{1}{2}$ , there exists  $p \in \mathbf{N}$  such that

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$$|a_{m+1} + a_{m+2} + \dots + a_n| < \frac{1}{2} \quad \text{for all } n \geq m \geq p, \quad (\text{i})$$

where  $a_n = \frac{1}{n}$

In particular, (i) is true for  $n = 2p$ ,  $m = p$  also.

That means

$$\left| \frac{1}{p+1} + \frac{1}{p+2} + \dots + \frac{1}{2p} \right| < \frac{1}{2} \quad (\text{ii})$$

Now,

$$\begin{aligned} \frac{1}{p+1} + \frac{1}{p+2} + \dots + \frac{1}{2p} &\geq \frac{1}{2p} + \frac{1}{2p} + \dots + \frac{1}{2p} \quad (\text{p-times}) \\ &= p \cdot \frac{1}{2p} = \frac{1}{2} \end{aligned}$$

Thus, we get a contradiction to (ii). Hence, our assumption is wrong. Therefore, the given series is divergent.

The Cauchy Criterion is a necessary as well as sufficient condition for the convergence of a series. Below we provide a necessary condition for the convergence of a series. It says that in a convergent series, it is the initial terms which mainly contribute to the sum of the series.

**Theorem 3.1.13.** If  $\sum_{n=1}^{\infty} a_n$  converges, then for  $\varepsilon > 0$ , there exists  $n_0 \in \mathbf{N}$  such that

$$\left| \sum_{k=n}^{\infty} a_k \right| < \varepsilon \quad \text{for any } n \geq n_0$$

**Proof.** Let  $S$  denote the sum and  $(S_n)$  denote the sequence of partial sums of the series

$$\sum_{n=1}^{\infty} a_n. \quad \text{Then, } \lim (S_n) = S.$$

Hence, for  $\varepsilon > 0$ , there exists  $m \in \mathbf{N}$  such that

$$|S_n - S| < \varepsilon \quad \text{for all } n \geq m$$

$$\Rightarrow |a_{n+1} + a_{n+2} + a_{n+3} + \dots| < \varepsilon \quad \text{for all } n \geq m \quad (\text{i})$$

We write,  $n_0 = m + 1$ . Then (i) can be rewritten as

$$\left| \sum_{k=n}^{\infty} a_k \right| < \varepsilon \quad \text{for all } n \geq n_0.$$

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The following example is an illustration of the above result.

**Example 3.1.14.** Consider the geometric series  $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ . It is a convergent series with

sum  $\frac{1}{1-\frac{1}{2}} = 2$ . It can be verified that

$$\left| \sum_{k=n}^{\infty} \frac{1}{2^{k-1}} \right| < .130 \quad \text{for all } n \geq 5$$

$$\left| \sum_{k=n}^{\infty} \frac{1}{2^{k-1}} \right| < .004 \quad \text{for all } n \geq 10$$

and  $\left| \sum_{k=n}^{\infty} \frac{1}{2^{k-1}} \right| < .000004 \quad \text{for all } n \geq 20$ .

### ILLUSTRATIVE EXAMPLES

1. Determine if the following series converge or diverge, using only the definition

(a)  $1-3+5-7+\dots+(-1)^{n+1}(2n-1)+\dots$

(b)  $1+1+1-1-1-1+1+1+1-1-1-1+\dots$

(c)  $\sum_{n=1}^{\infty} \left( \frac{n^2}{e^n} - \frac{(n+1)^2}{e^{n+1}} \right)$

(d)  $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$

**Solution.** (a) Here  $S_1 = 1, S_2 = 1-3 = -2, S_3 = -2+5 = 3, S_4 = 3-7 = -4, S_5 = -4+9 = 5, \dots$  and so on. Then, the sequence of partial sum here is

$$S_n = (1, -2, 3, -4, 5, -6, \dots, (-1)^{n+1}n, \dots)$$

Clearly,  $(S_n)$  is unbounded and therefore divergent.

Hence, the series is divergent.

(b) Here  $(S_n) = (1, 2, 3, 2, 1, 0, 1, 2, 3, 2, 1, 0, 1, 2, \dots)$  which is divergent.

Hence, the series is divergent.

(c) 
$$S_n = \left( \frac{1^2}{e^1} - \frac{2^2}{e^2} \right) + \left( \frac{2^2}{e^2} - \frac{3^2}{e^3} \right) + \dots + \left( \frac{n^2}{e^n} - \frac{(n+1)^2}{e^{n+1}} \right)$$

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$$= \frac{1}{e} - \frac{(n+1)^2}{e^{n+1}}$$

$$\lim(S_n) = \lim\left(\frac{1}{e} - \frac{(n+1)^2}{e^{n+1}}\right) = \frac{1}{e} - 0 = \frac{1}{e} \quad (\text{Using } \lim\left(\frac{n^2}{e^n}\right) = 0)$$

Thus,  $(S_n)$  is convergent.

Hence, the series is convergent.

$$(d) \quad a_n = \frac{1}{4n^2 - 1} = \frac{1}{(2n-1)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n+1}$$

$$\Rightarrow 1 = A(2n+1) + B(2n-1)$$

$$\Rightarrow A = \frac{1}{2}, B = -\frac{1}{2}$$

$$a_n = \frac{1}{2} \left( \frac{1}{2n-1} - \frac{1}{2n+1} \right)$$

$$S_n = a_1 + a_2 + a_3 + \dots + a_n$$

$$= \frac{1}{2} \left( 1 - \frac{1}{3} \right) + \frac{1}{2} \left( \frac{1}{3} - \frac{1}{5} \right) + \dots + \frac{1}{2} \left( \frac{1}{2n-1} - \frac{1}{2n+1} \right)$$

$$= \frac{1}{2} - \frac{1}{2(2n+1)}$$

$$\lim(S_n) = \frac{1}{2}$$

Thus,  $(S_n)$  is convergent. Hence, the given series is convergent.

2. Find the sum of each of following series, if it converges

(a)  $0.\overline{235}$

(b)  $\sum (-1)^n (.3)^{-2n}$

(c)  $\sum \frac{2^n + 3^n}{5^n}$

(d)  $\sum \frac{1}{(n+a)(n+a+1)}, a > 0$

(e)  $\sum \cos(n\pi)$

(f)  $1 - \frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \dots$

(g)  $(\ln 3) - (\ln 3)^2 + (\ln 3)^3 - \dots (m)$

(h)  $(\ln 2) + (\ln 2)^3 + (\ln 2)^5 + \dots$

**Solution.**

(a)  $0.\overline{235} = 0.235235235235\dots$

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$$= 0.235 + 0.000235 + 0.000000235 + \dots$$

$$= \frac{235}{10^3} \left( 1 + \frac{1}{10^3} + \frac{1}{10^6} + \frac{1}{10^9} + \dots \right)$$

This is an infinite G.P. series with  $r = \frac{1}{10^3} < 1$ .

Hence, it is convergent. Its sum  $S$  is given by

$$S = \frac{a}{1-r} = \frac{235/10^3}{1 - \frac{1}{10^3}} = \frac{235}{999}$$

(b)  $\sum (-1)^n (.3)^{-2n}$

Here the series is  $-\frac{1}{(.3)^2} + \frac{1}{(.3)^4} - \frac{1}{(.3)^6} - \dots$

This is a geometric series with  $r = -\frac{1}{(.3)^2}$ .

Now,  $|r| = \frac{1}{(.3)^2} = \frac{1}{.09} = \frac{100}{9} > 1$ .

Hence, the series is divergent.

(c) The given series is sum of two series  $\sum \left(\frac{2}{5}\right)^n$  and  $\sum \left(\frac{3}{5}\right)^n$ .

Since, sum of the two convergent series is convergent, the given series is convergent

and its sum is  $S = \frac{2/5}{1 - \frac{2}{5}} + \frac{3/5}{1 - \frac{3}{5}} = \frac{2}{3} + \frac{3}{2} = \frac{13}{6}$ .

(d)  $\sum \frac{1}{(n+a)(n+a+1)} = \sum \left( \frac{1}{n+a} - \frac{1}{n+a+1} \right)$

Here,  $S_n = \left( \frac{1}{1+a} - \frac{1}{2+a} \right) + \left( \frac{1}{2+a} - \frac{1}{3+a} \right) + \dots + \left( \frac{1}{n+a} - \frac{1}{n+1+a} \right)$

$$= \frac{1}{1+a} - \frac{1}{n+1+a}$$

$$\lim (S_n) = \frac{1}{1+a}$$

Therefore, the series is convergent and its sum is  $\frac{1}{1+a}$ .

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$$(e) \sum_{r=1}^{\infty} \cos n\pi = \cos \pi + \cos 2\pi + \cos 3\pi + \dots$$

$$= -1 + 1 - 1 + 1 - \dots$$

$$S_n = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Clearly  $(S_n)$  is not convergent. Hence, the series is divergent.

$$(f) \text{ Here } a_n = (-1)^{n+1} \frac{n-1}{n} = (-1)^{n+1} \left(1 - \frac{1}{n}\right), \text{ for } n \geq 2.$$

$\lim(a_n) \neq 0$  (In fact it does not exist).

Therefore, by the  $n^{\text{th}}$  term test, the series diverges.

(g) It is a G.P. series with  $|r| = \ln 3 > 1$ . Hence, it is divergent.

(h) It is a G.P. series with  $|r| = |(\ln 2)^2| < 1$ . Hence, it is convergent.

$$\text{The sum is, } S = \frac{a}{1-r} = \frac{\ln 2}{1 - (\ln 2)^2}.$$

3. Show that  $\sum_{n=1}^{\infty} \cos nx$  diverges for all  $x \in \mathbf{R}$ .

**Solution.**  $\lim_{n \rightarrow \infty} (\cos nx)$  does not exist for any nonzero real  $x$ . For  $x = 0$ ,  $\lim_{n \rightarrow \infty} \cos nx = 1$ .

Hence, by the  $n^{\text{th}}$  term test,  $\sum \cos nx$  is divergent for each  $x \in \mathbf{R}$ .

4. A series is called a *telescoping series* if it is of the form  $\sum_{n=1}^{\infty} (a_n - a_{n+1})$  or

$\sum_{n=1}^{\infty} (a_{n+1} - a_n)$ . Show that a telescoping series is convergent if and only if the sequence  $(a_n)$  converges. What is its sum ?

**Solution.** For the series  $\sum_{n=1}^{\infty} (a_n - a_{n+1})$ , we have

$$S_n = (a_1 - a_2) + (a_2 - a_3) + \dots + (a_n - a_{n+1})$$

$$= a_1 - a_{n+1}$$

$$\Rightarrow \lim(S_n) = a_1 - \lim(a_{n+1}).$$

Thus,  $\lim(S_n)$  exists if and only if  $\lim(a_{n+1})$  exists.

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In other words, the series converges if and only if the sequence  $(a_n)$  converges.

Also, when convergent, its sum

$$S = a_1 - \lim(a_{n+1}) = a_1 - \lim(a_n)$$

Similar is the case if the series is  $\sum(a_{n+1} - a_n)$ .

5. Find all  $x$  in  $[0, 2\pi]$  for which the series  $\sum (\sin x)(\cos x)^n$  converges.

**Solution.**  $\sum (\sin x)(\cos x)^n$  is a G.P. series with  $r = \cos x$ . Therefore, the series converges if and only if  $|\cos x| < 1$ .

We know that  $|\cos x| = 1$ , if  $x = n\pi$ ,  $n \in \mathbb{Z}$ .

Again for  $x = n\pi$  we have  $\sin x = 0$ . Therefore the given series converges for all values of  $x$  in  $[0, 2\pi]$ .

6. Show that any series can be written as a telescoping series.

**Solution.** Let  $\sum_{n=1}^{\infty} a_n$  be any given series. We write

$$S_n = a_1 + a_2 + a_3 + \dots + a_{n-1}, a_0 = 0.$$

Then,  $S_{n+1} - S_n = a_n$  so that  $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (S_{n+1} - S_n)$ .

7. Show that any sequence can be written as a sequence of partial sums of some series.

**Solution.** Let  $(a_n)$  be any sequence. We write

$$S_1 = a_1, S_2 = a_2 - a_1, S_3 = a_3 - a_2, \dots, S_n = a_n - a_{n-1}.$$

Then,  $a_1 = S_1, a_2 = S_1 + S_2, a_3 = a_2 + S_3 = S_1 + S_2 + S_3$ . Now, let

$$a_k = S_1 + S_2 + \dots + S_k$$

We have,  $S_n = a_n - a_{n-1}$  so that

$$S_{k+1} = a_{k+1} - a_k$$

Therefore

$$a_{k+1} = a_k + S_{k+1} = S_1 + S_2 + \dots + S_k + S_{k+1}$$

Applying the Principle of Mathematical Induction, we get

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$$a_n = \sum_{k=1}^n S_k \quad \text{for all } n \in \mathbf{N}.$$

8. If  $\sum |a_n|$  is convergent and  $(b_n)$  is bounded, then show that  $\sum a_n b_n$  is convergent.

**Solution.** Since  $(b_n)$  is bounded, there exists  $k > 0$  such that  $|b_n| \leq k$  for all  $n \in \mathbf{N}$ .

Let  $\varepsilon > 0$  be given. As  $\sum |a_n|$  is convergent, by the Cauchy's Criterion, there exists  $p \in \mathbf{N}$  such that

$$\|a_{m+1}| + |a_{m+2}| + \dots + |a_n|\| < \frac{\varepsilon}{k} \quad \text{for all } n \geq m \geq p$$

$$\begin{aligned} \text{Now } & |a_{m+1}b_{m+1} + a_{m+2}b_{m+2} + \dots + a_n b_n| \\ & \leq |a_{m+1}b_{m+1}| + |a_{m+2}b_{m+2}| + \dots + |a_n b_n| \\ & = |a_{m+1}||b_{m+1}| + |a_{m+2}||b_{m+2}| + \dots + |a_n||b_n| \\ & \leq k(|a_{m+1}| + |a_{m+2}| + \dots + |a_n|) \\ & < k \cdot \frac{\varepsilon}{k} \quad \text{for all } n \geq m \geq p \\ & = \varepsilon \quad \text{for all } n \geq m \geq p \end{aligned}$$

Hence, by the Cauchy Criterion,  $\sum a_n b_n$  is convergent.

9. Find a rational number that is limit of the repeating decimal  $0.\overline{15}$ .

**Solution.**  $0.\overline{15} = 0.15151515\dots$

$$\begin{aligned} & = 0.15 + 0.0015 + 0.000015 + \dots \\ & = 0.15(1 + 0.01 + 0.0001 + \dots) \\ & = 0.15(1 + 0.01 + (0.01)^2 + \dots) \\ & = \sum_{n=1}^{\infty} 0.15(0.01)^{n-1} \end{aligned}$$

This is a G.P. series with  $|r| = .01 < 1$ . Hence, it is convergent and its sum is

$$S = \frac{a}{1-r} = \frac{.15}{1-.01} = \frac{15}{99} = \frac{5}{33}.$$

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10. Prove that the series  $\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{9} + \frac{1}{8} - \frac{1}{27} + \dots + \frac{1}{2^n} - \frac{1}{3^n} + \dots$  converge to  $\frac{1}{2}$ .

**Solution.** The given series is of the form  $s_1 + s_2 + s_3 + \dots +$  where  $s_1 = \frac{1}{2} - \frac{1}{3}$ ,  $s_2 = \frac{1}{4} - \frac{1}{9}$  ... We write  $a_n = \frac{1}{2^n}$  and  $b_n = -\frac{1}{3^n}$ . Then, the series  $\sum a_n = \sum \frac{1}{2^n}$  and  $\sum b_n = \sum (-1) \frac{1}{3^n}$  are geometric series with  $r = \frac{1}{2}$  and  $r = \frac{1}{3}$  respectively.

Hence, both are convergent. Therefore  $\sum (a_n + b_n)$  is also convergent.

$$\begin{aligned}\text{Also, } \sum (a_n + b_n) &= \sum a_n + \sum b_n \\ &= \frac{1/2}{1 - \frac{1}{2}} + \frac{-1/3}{1 - \frac{1}{3}} \quad \left( \sum ar^{n-1} = \frac{a}{1-r} \right) \\ &= 1 - \frac{1}{2} = \frac{1}{2}.\end{aligned}$$

### EXERCISES 3.1

1. Determine if the following series are convergent or divergent, using only the definition or the  $n^{\text{th}}$  term test

(a)  $2 - 4 + 6 - 8 + 10 - 12 + \dots$

(b)  $2 + 3 + 5 - 2 - 3 - 5 + 2 + 3 + 5 - 2 - 3 - 5$

(c)  $\sum_{n=1}^{\infty} \left( \frac{n^2}{\pi^n} - \frac{(n+1)^2}{\pi^{n+1}} \right)$

(d)  $\sum \frac{1}{4n^2 - 1}$

(e)  $\sum \frac{1}{n(n+1)}$

(f)  $\sqrt{\frac{1}{4}} + \sqrt{\frac{2}{6}} + \sqrt{\frac{3}{8}} + \dots + \sqrt{\frac{n}{2(n+1)}} + \dots$

(g)  $1 - \frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \dots$

2. Find the sum of each of the following series, if it converges

(a)  $0.\overline{413}$

(b)  $5.\overline{315}$

(c)  $\sum (-1)^n (.2)^{-4n}$

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$$(d) \sum (-1)^n (.1)^{4n} \quad (e) \sum \sin\left(\frac{n\pi}{2}\right) \quad (f) \sum \frac{1}{n^2 + 5n + 6}$$

$$(g) \sum \frac{n-1}{2^{n+1}} \quad (h) \sum \frac{n}{2^n} \quad (i) \sum \sin\left(\frac{n\pi}{4}\right)$$

$$(j) \sum \left[ \frac{2}{3 + (-1)^n} \right]^n \quad (k) \sum \left( \frac{2}{3^n} + \frac{5}{n^2 + n} \right) \quad (l) \sum_{n \geq 2} \ln\left(\frac{n}{n+1}\right)$$

$$(m) \sum \frac{2n}{3n+5} \quad (n) \sum \sin(n\pi) \quad (o) \sum \frac{7}{10^n}$$

$$(p) \sum (-1)^n (.3)^{2n} \quad (q) \sum \left(\frac{\pi}{e}\right)^n$$

3. Test for convergence of the series whose  $n^{\text{th}}$  term is

$$(a) \cos\left(\frac{1}{n}\right) \quad (b) \cot^{-1}\left(\frac{1}{n}\right) \quad (c) (-1)^n a^{1/n}, a > 0$$

$$(d) n \sin^{-1}\left(\frac{1}{n}\right) \quad (e) \frac{n}{1+2^{-n}} \quad (f) \left(\frac{1}{n}\right)^{1/n}$$

$$(g) \left(1 - \frac{1}{n}\right)^{1/n} \quad (h) n \tan^{-1}\left(\frac{1}{n}\right) \quad (i) \left[1 + \left(\frac{a}{n}\right)\right]^{bn+c}$$

$$(j) n^2 \sin \frac{3}{n} \tan \frac{5}{n} \quad (k) \cos\left(\frac{n\pi}{2}\right) \quad (l) \frac{n}{\sqrt{n^2+2}}$$

$$(m) \sin\left(\frac{\pi}{2} - \frac{1}{n}\right) \quad (n) e^{nx}$$

4. Show that the series  $1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} + \dots$  is convergent.

5. Show that the series  $1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2n-1} + \dots$  is divergent.

6. Show that the series  $1 + \frac{1}{4} + \frac{1}{7} + \dots + \frac{1}{3n-2} + \dots$  is divergent.

7. Write an infinite series for the repeating decimal for the rational number  $\frac{5}{9}$  and prove that it converges to  $\frac{5}{9}$ .

8. Show that if  $\frac{a_{n+1}}{a_n}$  is a constant, then  $\sum a_n$  is a G.P. series.

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9. Show that if  $\Sigma a_n$  converges, then  $-1 < a_n < 1$  for all  $n \in \mathbf{N}$ , except for finitely many values of  $n$ .
10. Prove that a necessary and sufficient condition for the series  $\Sigma a_n$  to converge is that  $\lim(R_n) = 0$ , where  $R_n = a_{n+1} + a_{n+2} + \dots$
11. Show that if  $\Sigma a_n$  converges, so does  $\Sigma(a_n + a_{n+1})$ .
12. If  $\Sigma a_n$  is a divergent series of positive decreasing terms, then show that

$$\lim_{n \rightarrow \infty} \frac{a_2 + a_4 + \dots + a_{2n}}{a_1 + a_3 + \dots + a_{2n-1}} = 1.$$

$$\lim \left( \frac{a_n}{b_n} \right) = 0 \text{ and } \Sigma a_n \text{ converges ?}$$