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3.5 ALTERNATING SERIES AND ARBITRARY TERM SERIES

So far we have discussed various tests of convergence for the series of positive terms. Now, we introduce series consisting of positive and negative terms alternately. Such series are called *alternating series*.

Definition 3.5.1. A series of the form $\sum(-1)^{n+1}a_n$ or $\sum(-1)^n a_n$ with $a_n > 0$ is called an *alternating series*.

Examples of alternating series are

(i) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-...$

(ii) $-1+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-...$

(iii) $2-2^2+2^3-2^4+2^5-...$

(iv) $1-\frac{1}{2}+3-\frac{1}{4}+5-\frac{1}{6}+...$ etc.

Below we discuss a test called 'Alternating Series test' or 'Leibnitz's test' to determine convergence of an alternating series.

Theorem (Alternating Series Test) 3.5.2. The alternating series $\sum(-1)^{n+1}a_n$ or $\sum(-1)^n a_n$ converges if the following two conditions hold :

(i) $a_{n+1} \le a_n$ for all n;

(ii) $\lim(a_n) = 0$

Proof. We provide the proof for $\sum(-1)^{n+1}a_n$ only as the proof for $\sum(-1)^n a_n$ is essentially on the similar lines.

We denote the sequence of partial sums of $\sum(-1)^{n+1}a_n$ by (S_n). Then, for any n ∈ **N**.

$$S_{2n+2} = S_{2n} + (a_{2n+1} - a_{2n+2})$$

and $$S_{2n+1} = S_{2n-1} + (-a_{2n} + a_{2n+1})$$

By condition (i), we have, $a_{2n+1} - a_{2n+2} \geq 0$ and $-a_{2n} + a_{2n+1} \leq 0$.

Thus, $S_{2n+2} \geq S_{2n}$ and $S_{2n+1} \leq S_{2n-1}$.

Therefore the subsequences (S_{2n}) and (S_{2n-1}) of (S_n) are increasing and decreasing respectively. Again $S_{2n} - S_{2n-1} = -a_{2n} < 0$. Hence, $S_{2n} \leq S_{2n-1}$.

Thus, for any n ∈ **N**, we have

$$S_2 \leq S_{2n} \leq S_{2n-1} \leq S_1.$$

Accordingly, both (S_{2n}) and (S_{2n-1}) are monotone and bounded and hence are convergent, by the monotone convergent theorem. From condition (ii) of the hypothesis, that is, $\lim(a_n) = 0$ and the fact that $S_{2n} = S_{2n-1} - a_{2n}$, it follows that both (S_{2n}) and (S_{2n-1}) converge to the same limit, say S.

Now, we show that (S_n) converges to S. Let $\varepsilon > 0$ be given.

As $\lim(S_{2n}) = S = \lim(S_{2n-1})$, there exist positive integers m_1, m_2 such that

$$|S_{2n} - S| < \varepsilon \qquad \text{for all } n \geq m_1$$

and $$|S_{2n-1} - S| < \varepsilon \qquad \text{for all } n \geq m_2$$

We write $m = \max\{2m_1, 2m_2 - 1\}$. Then clearly, we have

$$|S_n - S| < \varepsilon \qquad \text{for all } n \geq m$$

Hence, (S_n) converges to S.

Therefore, the series $\sum(-1)^{n+1}a_n$ is convergent.

Corollary 3.5.3. Under the hypothesis of the above theorem, we have

$$|S - S_n| \leq a_{n+1} \qquad \text{for any } n \in \mathbf{N}.$$

Proof. Since (S_{2n}) is an increasing sequence, we have

$$S = \lim(S_{2n}) = \sup\{S_{2n} : n \in \mathbf{N}\}.$$

Therefore, $S_{2n} \leq S$. Similarly, $S_{2n+1} \geq S$.

Now, $$S_{2n} = S_{2n-1} - a_{2n}$$

and $\quad S_{2n+1} = S_{2n} + a_{2n+1}$

Thus, $\quad S_{2n-1} - a_{2n} \le S \le S_{2n} + a_{2n+1}$

That is, $\quad |S_{2n-1} - S| \le a_{2n}$ and $|S - S_{2n}| \le a_{2n+1}$.

Combining them we get

$$|S - S_n| \le a_{n+1} \text{ for any } n \in \mathbf{N}$$

Remark. The conditions in the Leibnitz's test are sufficient however not necessary. Therefore, if the conditions are not satisfied, we cannot say that the alternating series is divergent on the basis of Leibnitz's test

Example 3.5.4. (i) The series $\sum (-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ is convergent as $\frac{1}{n+1} \le \frac{1}{n}$ for every $n \in \mathbf{N}$ and $\lim\left(\frac{1}{n}\right) = 0$

(ii) The series $\sum (-1)^{n+1}\frac{1}{n^2}$ is convergent.

(iii) We can not comment on the convergence of the series $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \ldots$, using Alternating series test. Although $\lim(a_n) = 0$ here, but $a_{n+1} \le a_n$ does not hold.

(iv) The series $\sum (-1)^{n+1}\frac{n}{e^n}$ is convergent. Here $\lim(a_n) = \lim\left(\frac{n}{e^n}\right) = 0$. That $a_{n+1} \le a_n$ can be ascertained by the fact that for $f(x) = \frac{x}{e^x}$, $x \ge 1$, we have $f'(x) = \frac{1-x}{e^x} \le 0$. That is, $f(x)$ is decreasing for $x \ge 1$.

Absolute Convergence

Look at the series $\sum (-1)^{n+1}\frac{1}{n}$ and $\sum (-1)^{n+1}\frac{1}{n^2}$ discussed above.

Both these series are convergent. However, if we ignore their negative signs, then we get the series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$ respectively. The first series is divergent while the second one is convergent. This leads to the concept of conditional convergence and absolute convergence of series of arbitrary terms.

Definition 3.5.5. Let $\sum a_n$ be a series of arbitrary terms (that is, $a_n \ge 0$ or $a_n < 0$ for any n). We say

(i) $\sum a_n$ *converges absolutely* if $\sum |a_n|$ converges;

(ii) $\sum a_n$ *converges conditionally* if $\sum a_n$ converges but $\sum |a_n|$ diverges.

For example, the alternating series $\sum (-1)^n \frac{1}{n}$ is conditionally convergent, while $\sum (-1)^{n+1} \frac{1}{n^2}$ is absolutely convergent.

That absolute convergence is stronger than convergence is clear from the following result.

Theorem 3.5.6. If the series $\sum |a_n|$ converges, then so does $\sum a_n$ and $\left|\sum a_n\right| \le \sum |a_n|$.

Proof. Since $\sum |a_n|$ converges, therefore, given any ε > 0, by the Cauchy's convergence criterion on series, there exists p ∈ **N** such that

$$|a_{m+1}| + |a_{m+2}| + \ldots + |a_n| < \varepsilon \quad \text{for all } n \ge m \ge p.$$

Now, using the triangular inequality, we get

$$\begin{aligned} |a_{m+1} + a_{m+2} + \ldots + a_n| &\le |a_{m+1}| + |a_{m+2}| + \ldots + |a_n| \\ &< \varepsilon \qquad \text{for all } n \ge m \ge p \end{aligned}$$

Thus, $\sum a_n$ satisfies the Cauchy's convergence criterion and hence is convergent.

Now for any m ∈ **N**,

$$\left|\sum_{n=1}^{m} a_n\right| \le \sum_{n=1}^{m} |a_n|, \quad \text{by triangular inequality.}$$

Letting m → ∞ yields the desired inequality $\left|\sum a_n\right| \le \sum |a_n|$ of the sums of the two series.

Example 3.5.7. (i) The alternating series $\sum (-1)^{n+1} \frac{1}{n}$ converges conditionally while $\sum (-1)^{n+1} \frac{1}{n^2}$ converges absolutely.

(ii) The series $\sum (-1)^{n+1} \frac{n}{e^n}$ is absolutely convergent (see Example 3.5.4).

In fact, if we take $a_n = \frac{n}{e^n}$ and $b_n = \frac{1}{2^n}$, then

$$\lim\left(\frac{a_n}{b_n}\right) = \lim n\left(\frac{2^n}{e^n}\right) = \lim n\left(\frac{2}{e}\right)^n = 0$$

[Since, $\lim(nx^n) = 0$ if $0 < x < 1$, by the ratio test for sequence]

Since the G.P. series $\sum \frac{1}{2^n}$ is convergent, $\sum \frac{n}{e^n}$ is convergent by Corollary 3.2.4.

Hence, $\sum (-1)^{n+1} \frac{n}{e^n}$ is absolutely convergent.

(iii) The series $1 - \frac{1}{\sqrt[3]{2}} - \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{5}} - \frac{1}{\sqrt[3]{6}} - \ldots$ is conditionally convergent.

Example 3.5.8. Test for convergence the following series

(i) $\sum (-1)^n \frac{\cos n\pi}{n}$ (ii) $\sum (-1)^n \frac{\sin n}{n^{3/2}}$ (iii) $\sum \frac{(-1)^n - (-1)^{n+1}}{n+1}$

(iv) $\sum (-1)^{n+1} \frac{n}{n+1}$

Solution.

(i) Since $\cos n\pi = (-1)^n$, we have

$$\sum (-1)^n \frac{\cos n\pi}{n} = \sum \frac{1}{n} \text{ which is divergent by p-test}$$

(ii) We have, $\left|(-1)^n \frac{\sin n}{n^{3/2}}\right| \le \frac{1}{n^{3/2}}$ for all n.

Since $\sum \frac{1}{n^{3/2}}$ is convergent, $\sum (-1)^n \frac{\sin n}{n^{3/2}}$ is absolutely convergent.

(iii) $\sum \frac{(-1)^n - (-1)^{n+1}}{n+1} = \sum (-1)^n \frac{2}{n+1}$.

The given series is an alternating series, with $a_{n+1} \le a_n$ for all n and $\lim(a_n) = 0$.

Hence, by the alternating series test, it is converget.

However, $\sum \frac{1}{n+1}$ is divergent (check it).

Therefore, the given series is conditionally convergent.

(iv) Here $\lim\left(\frac{n}{n+1}\right) = 1 \ne 0$, so that $\lim\left((-1)^n \frac{n}{n+1}\right) \ne 0$.

Hence, by the n^{th} term test, the series is divergent.

ILLUSTRATIVE EXAMPLES

1. For each of the following series, determine whether the series converges absolutely, converges conditionally or diverges

(a) $\sum(-1)^n \frac{n}{2n-1}$

(b) $\sum(-1)^n \frac{n}{n^2+1}$

(c) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+(-1)^n}$

(d) $\sum(-1)^n e^{-n}$

(e) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

(f) $\sum(-1)^n \frac{\arctan n}{n}$

(g) $\sum(-1)^n ne^{-n}$

Solution. (a)We write, $a_n = \frac{n}{2n-1}$. Then, $\lim(a_n) = \frac{1}{2} \neq 0$.

Hence, the series diverges, by the n^{th} term test.

(b) Let $a_n = \frac{n}{n^2+1}$. We take $b_n = \frac{1}{n}$.

Then, $\frac{a_n}{b_n} = \frac{n}{n^2+1}.\frac{n}{1} = \frac{1}{1+\frac{1}{n^2}}$

$$\lim\left(\frac{a_n}{b_n}\right) = 1.$$

Hence, by the limit comparison test $\sum\frac{n}{n^2+1}$ diverges as $\sum\frac{1}{n}$ is divergent.

Now, we check for conditional convergence.

$$a_{n+1} - a_n = \frac{n+1}{(n+1)^2+1} - \frac{n}{n^2+1}$$

$$= \frac{(n^3+n^2+n+1)-(n^3+2n^2+2n)}{((n+1)^2+1)(n^2+1)}$$

$$= \frac{-n^2-n+1}{((n+2)^2+1)(n^2+1)} < 0$$

Hence, $a_{n+1} \leq a_n$ for all n.

Also, $\lim(a_n) = \lim\left(\frac{n}{n^2+1}\right) = \lim\left(\frac{1}{n+\frac{1}{n}}\right) = 0$

Hence, by Leibnitz's test, the series $\sum(-1)^n \frac{n}{n^2+1}$ converges conditionally.

(c) We write, $a_n = \frac{1}{n^2+(-1)^n}$. Take $b_n = \frac{1}{n^2}$.

Then, $\frac{a_n}{b_n} = \frac{n^2}{n^2+(-1)^n} = \frac{1}{1+\frac{(-1)^n}{n^2}}$

$$\lim\left(\frac{a_n}{b_n}\right)=1.$$

Hence, by the Limit Comparison Test $\sum a_n$ converges as $\sum \frac{1}{n^2}$ is a convergent series by the p-test.

Hence, $\sum(-1)^n a_n = \sum \frac{(-1)^n}{n^2+(-1)^n}$ is absolutely convergent.

Rest of the parts can be done in a similar manner and are left as exercises.

2. Discuss the convergence of the following series

(a) $\sum(-1)^{n+1}\frac{\ln n}{n}$ (b) $\sum(-1)^{n+1}\frac{(\ln n)^2}{n}$

(c) $\sum(-1)^{n+1}\frac{(\ln n)^p}{n}$, $p > 0$

Solution. All of these series are alternating series.

For $n \geq 3$, we have $\ln n \geq 1$. Hence, we have

$$\frac{\ln n}{n} \geq \frac{1}{n},\ \frac{(\ln n)^2}{n} \geq \frac{1}{n},\ \frac{(\ln n)^p}{n} \geq \frac{1}{n}.$$

Thus, all of $\sum\frac{\ln n}{n}$, $\sum\frac{(\ln n)^2}{n}$ and $\sum\frac{(\ln n)^p}{n}$ are divergent by the comparison test.

Hence, we need to check these series for conditional convergence.

(a) We write, $f(x) = \frac{\ln x}{x}$, then

$$f'(x) = \frac{1-\ln x}{x^2} < 0 \quad \text{for } x > e$$

Hence, $\left(\frac{\ln n}{n}\right)$ is decreasing for $n \geq 3$.

Again, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{\ln x}{x} \quad \left(\frac{\infty}{\infty}\text{form}\right)$

$$= \lim_{x\to\infty}\frac{1}{x} \qquad \text{(L'Hospital's rule)}$$

$$= 0$$

Hence, $\lim\left(\frac{\ln n}{n}\right) = 0$

Hence, by alternating series test, $\sum(-1)^{n+1}\frac{\ln n}{n}$ is conditionally convergent.

(b) and (c) parts can be done similarly.

3. Show that the following series is convergent

$$\frac{\ln 2}{2^2}-\frac{\ln 3}{3^2}+\frac{\ln 4}{4^2}-\frac{\ln 5}{5^2}+\ldots$$

Solution. This is an alternating series $\sum(-1)^{n+1}a_n$, where $a_n=\frac{\ln(n+1)}{(n+1)^2}$

$$\lim(a_n)=\lim\left(\frac{\ln(n+1)}{n+1}\right)\lim\left(\frac{1}{n+1}\right)=0 \qquad \left[\text{Using } \lim\left(\frac{\ln n}{n}\right)=0\right]$$

To show that (a_n) is decreasing, we write

$$f(x)=\frac{\ln x}{x^2}, \quad x\geq 1$$

$$f'(x)=\frac{x-2x\ln x}{x^4}=\frac{1-2\ln x}{x^3}$$

Thus, $f'(x)<0$ if $2\ln x>1$, that is, if $x>e^{1/2}$.

Hence, f(x) is monotonically decreasing for all $x>e^{1/2}$.

Therefore, $a_n \leq a_{n+1}$ for all $n \geq 2$.

Hence, by the alternating series test, the given series is covnvergent.

4. If Σa_n and Σb_n converge absolutely, then show that

(a) $\Sigma(a_n + b_n)$ converge absolutely

(b) Σa_n^2 converges

Solution. (a) $\Sigma|a_n|$, $\Sigma|b_n|$ and $\Sigma|a_n + b_n|$ are all positive term series with

$$|a_n+b_n|\leq|a_n|+|b_n| \qquad \text{(Triangular Inequality)} \qquad \text{(i)}$$

Now, Σa_n, Σb_n converge absolutely.

$\Rightarrow$ $\Sigma|a_n|$, $\Sigma|b_n|$ converge

$\Rightarrow$ $\Sigma(|a_n| + |b_n|)$ is convergent, as sum of convergent series is convergent.

Hence, by comparison test, $\Sigma|a_n + b_n|$ also converges, in view of (i).

(b) $a_n^2=|a_n|^2$

Now, $\Sigma|a_n|$ is convergent and $\Sigma a_n^2=\Sigma|a_n|^2$.

Hence, Σa_n^2 is convergent (see Illustrative Example 2 of Section 3.2).

5. Show that if Σa_n converges and Σb_n converges absolutely then $\Sigma a_n b_n$ converges absolutely

Solution. As Σa_n is convergent, (a_n) converges to 0. Since, every convergent sequence is bounded, (a_n) is bounded. Thus, there exists $k > 0$ such that

$$|a_n| \le k \quad \text{for all n} \qquad \text{(i)}$$

Since, Σb_n converges absolutely, $\Sigma|b_n|$ satisfies Cauchy's convergence criterion.

Hence for a given $\varepsilon > 0$, there exist $p \in \mathbf{N}$ such that

$$\left||b_{m+1}| + |b_{m+2}| + \ldots + |b_n|\right| < \frac{\varepsilon}{k} \quad \text{for all } n \ge m \ge p.$$

Now, $\left||a_{m+1}b_{m+1}| + |a_{m+2}b_{m+2}| + \ldots + |a_n b_n|\right|$

$$= |a_{m+1}||b_{m+1}| + |a_{m+2}||b_{m+2}| + \ldots + |a_n||b_n|$$

$$\le k(|b_{m+1}| + |b_{m+2}| + \ldots + |b_n|) \qquad \text{(from (i))}$$

$$< k.\frac{\varepsilon}{k} = \varepsilon \quad \text{for all } n \ge m \ge p.$$

Hence, $\sum |a_n b_n|$ is convergent by Cauchy's criterion.

That is, $\sum a_n b_n$ converges absolutely.

6. If $\left|\frac{a_{n+1}}{a_n}\right| \le \frac{n^2}{(n+1)^2}$ for all $n \ge 1$, prove that the series Σa_n converges absolutely.

Solution. $|a_n| = \left|\frac{a_n}{a_{n-1}}\right| \cdot \left|\frac{a_{n-1}}{a_{n-2}}\right| \ldots \left|\frac{a_2}{a_1}\right| |a_1|$

$$\le \frac{(n-1)^2}{n^2} \cdot \frac{(n-2)^2}{(n-1)^2} \ldots \frac{1^2}{2^2} |a_1|, \text{ by the given condition}$$

$$= \frac{|a_1|}{n^2}.$$

Thus, $|a_n| \le \frac{|a_1|}{n^2}$ for all n.

Now, the series $\Sigma \frac{1}{n^2}$ is convergent by p-test. Hence, $\sum \frac{|a_1|}{n^2}$ is convergent. Therefore, by the Comparison Test, $\Sigma|a_n|$ is convergent. That is, Σa_n is absolutely convergent.

7. Discuss the convergence of

$$1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{4} + \frac{1}{4^2} - \ldots - \frac{1}{n} + \frac{1}{n^2} - \ldots$$

Solution. $\sum a_n^+ = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$, which is convergent.

$$\sum a_n^- = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots + \frac{1}{n+1} + \ldots, \text{ which is divergent}$$

Hence, the series is divergent.

EXERCISES 3.5

1. Determine convergence (absolute or conditional) of the following series

(a) $\sum \dfrac{(-1)^{n+1}}{3n+1}$

(b) $\sum \dfrac{(-1)^{n+1}}{n(n-3)}$

(c) $\sum \dfrac{(-1)^{n+1}}{\ln n}$

(d) $\sum (-1)^{n-1} \dfrac{n^2}{(n+1)!}$

(e) $\sum (-1)^n \dfrac{\sin n\alpha}{n^3}$, α being real

(f) $\sum (-1)^n \dfrac{\cos n\alpha}{n\sqrt{n}}$, α being real

(g) $\sum \dfrac{\sin n}{n^2+4}$

(h) $\sum\limits_{n=2}^{\infty} \dfrac{(-1)^{n+1}}{(\ln n)^n}$

(i) $\sum (-1)^{n+1} \left[\dfrac{1}{n^2} + \dfrac{1}{(n+1)^2}\right]$

(j) $\sum \left[\dfrac{1}{n} + \dfrac{(-1)^{n-1}}{\sqrt{n}}\right]$

2. Discuss the convergence of the following series

(a) $\dfrac{1}{2} - \dfrac{1}{4} + \dfrac{1}{6} - \dfrac{1}{8} + \ldots$

(b) $\dfrac{1}{\sqrt{1}} - \dfrac{1}{\sqrt{3}} + \dfrac{1}{\sqrt{5}} - \dfrac{1}{\sqrt{7}} + \ldots$

(c) $\dfrac{1}{2+\sqrt{2}} - \dfrac{1}{3+\sqrt{3}} + \dfrac{1}{4+\sqrt{4}} - \dfrac{1}{5+\sqrt{5}} + \ldots$

(d) $\dfrac{1}{2(\ln 2)^p} - \dfrac{1}{3(\ln 3)^p} + \dfrac{1}{4(\ln 4)^p} - \ldots$

(e) $\sum (-1)^{n+1} \dfrac{x^{2n+1}}{(2n+1)!}$

3. Show that $\sum \dfrac{(-1)^{n+1}}{n^p}$ is absolutely convergent for $p > 1$ and conditionally convergent for $0 < p \leq 1$.

4. Show that the following series converges if and only if $-1 < x \leq 1$.

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + (-1)^{n-1} \frac{x^{2n-1}}{2n-1} + \ldots$$

5. Show that the series

$$\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{9} + \frac{1}{8} - \frac{1}{27} + \ldots + \frac{1}{2^n} - \frac{1}{3^n} + \ldots$$

converges. Does it converge absolutely or conditionally ?

6. Does the series

$$\frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{3^2} + \frac{1}{4} - \frac{1}{3^3} + \frac{1}{5} - \frac{1}{3^4} + \ldots + \frac{1}{n} - \frac{1}{3^{n-1}} + \ldots$$

converge ? Justify your answer.

7. Check for convergence (conditional or absolute) the following series

(a) $\sum_{n=1}^{\infty}\left(\frac{1}{2n+1}-\frac{1}{2^n}\right)$

(b) $\sum_{n=1}^{\infty}\left(\frac{1}{5^n}-\frac{1}{3^n}\right)$

(c) $\frac{1}{3}-\frac{1}{5^2}+\frac{1}{3^2}-\frac{1}{9^2}+\frac{1}{3^3}-\frac{1}{13^2}+\ldots$

(d) $\frac{1}{\ln 2}-\frac{1}{2^2}+\frac{1}{\ln 3}-\frac{1}{2^3}+\frac{1}{\ln 4}-\frac{1}{2^4}+\ldots$

(e) $\frac{1}{\ln 2}-\frac{1}{\ln 3}+\frac{1}{\ln 4}-\frac{1}{\ln 5}+\frac{1}{\ln 6}-\frac{1}{\ln 7}+\ldots$