Since the convergence of a series is defined in terms of the convergence of its sequence of partial sums, Cauchy's Criterion for convergence of sequence can be adopted for series as well. Here is the reShow that if Σa_n converges, then $-1 < a_n < 1$ for all n $\in$ **N**, except f

If a sequence is not convergent, the next possibility is that the sequence may have convergent subsequences. In fact, the Bolzano–Weierstrass Theorem ensures this for a bounded sequence. Thus, the set of the subseqeuntial limits of a bounded sequence is never an empty set. The supremum and infimum of the set of the subsequential limits of a bounded sequence are of special interest in analysis – they are the limit supreior and limit inferior respectively of the sequence. However, to begin with, the definitions of limit suprerior and limit inferior donot involve subsequential limit, rather they are based on the range set of the sequence. We have the following definitions

Definition 2.8.1. (i) Let (x_n) be a sequence of real numbers which is bounded above. For each n $\in$ **N,** we define

$$\overline{x}_n = \sup\{x_n, x_{n+1}, x_{n+2}, ...\}$$

Then the *limit superior* of (x_n), denoted by lim sup (x_n), is defined to be the limit of the sequence $(\overline{x}_n)$. That is,

$$\begin{aligned} \lim\sup(x_n) &= \lim(\overline{x}_n) \\ &= \lim\left(\sup\{x_n, x_{n+1}, x_{n+2}, ...\}\right) \end{aligned}$$

Since, (x_n) is bounded above, each $\overline{x}_n$ is real. Also,

$$\overline{x}_1 \geq \overline{x}_2 \geq \overline{x}_3 \geq ... \geq \overline{x}_n \geq ...$$

Hence, $\lim(\overline{x}_n)$ is either a real number or $-\infty$.

(ii) If (x_n) is not bounded above, then we define $\overline{x}_n = +\infty$ for each n $\in$ **N** and

$$\lim\sup(x_n) = +\infty.$$

Definition 2.8.2. (i) Let (x_n) be any sequence of real numbers which is bounded below. For each n $\in$ **N**, we define

$$\underline{x_n} = \inf\{x_n, x_{n+1}, x_{n+2}, ...\}$$

Then, the *limit inferior* of (x_n), denoted by lim inf(x_n), is defined to be the limit of the sequence $(\underline{x_n})$. That is

$$\begin{aligned} \lim\inf(x_n) &= \lim(\underline{x_n}) \\ &= \lim(\inf\{x_n, x_{n+1}, x_{n+2}, ...\}) \end{aligned}$$

Since, (x_n) is bounded below, each $\underline{x}_n$ is a real number.

Also, $\underline{x}_1 \le \underline{x}_2 \le \underline{x}_3 \le \ldots \le \underline{x}_n \le \ldots$

Hence, $\lim (\underline{x}_n)$ is either real or $+\infty$.

(ii) If (x_n) is not bounded below, then we define

$$\underline{x}_n = -\infty \qquad \text{for each } n \in \mathbf{N}$$

and $\liminf (x_n) = -\infty$.

Example 2.8.3. Find the limit superior and limit inferior of the following seqeunces

(a) $(x_n) = ((-1)^n)$

(b) $(y_n) = \left(1, 2, \frac{1}{3}, 4, 5, \frac{1}{6}, 7, 8, \frac{1}{9}, \ldots.\right)$

(c) $(z_n) = \left(\frac{(-1)^{n+1}}{n}\right)$

(d) $(w_n) = (-n)$

Solution. (a) $(x_n) = (-1, 1, -1, 1, \ldots.)$. For any $n \in \mathbf{N}$

$$\overline{x}_n = \sup\{x_n,\ x_{n+1}, x_{n+2}, \ldots\} = 1$$

$$\underline{x}_n = \inf\{x_n, x_{n+1}, x_{n+2}, \ldots\} = -1$$

Hence, $\limsup (x_n) = 1$

and $\liminf (x_n) = -1$

(b) $(y_n) = \left(1,\ 2,\ \frac{1}{3},\ 4,\ 5,\ \frac{1}{6},\ 7, 8,\ \frac{1}{9}, \ldots\right)$

Here, (y_n) is not bounded above. Accordingly, $\overline{y}_n = +\infty$ and $\limsup (y_n) = +\infty$.

$\underline{y}_n = \inf\{y_n, y_{n+1}, \ldots\}$.

Hence, $\underline{y}_n = 0$, for any $n \in \mathbf{N}$,

so $\liminf (y_n) = 0$.

(c) $(z_n) = \left(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots\right)$

$$\overline{z}_n = \sup\left\{\frac{(-1)^{n+1}}{n},\ \frac{(-1)^{n+2}}{n+1}, \ldots\right\}$$

$$= \begin{cases} \frac{1}{n}, & \text{if } n \text{ is odd} \\ \frac{1}{n+1}, & \text{if } n \text{ is even} \end{cases}$$

Therefore, $\lim\sup(z_n) = \lim(\overline{z}_n) = 0$

Similarly,

$$\underline{z_n} = \begin{cases} -\frac{1}{n+1}, & \text{if } n \text{ is odd;} \\ -\frac{1}{n}, & \text{if } n \text{ is even.} \end{cases}$$

Hence, $\lim\inf(z_n) = \lim(\underline{z_n}) = 0$

(d) $(w_n) = (-1, -2, -3, -4, \ldots, -n, \ldots)$

$$\overline{w}_n = \sup\{-n, -(n+1), -(n+2), \ldots\} = -n$$

$$\underline{w_n} = \inf\{-n, -(n+1), -(n+2), \ldots\}$$

$$= -\infty, \qquad \text{as } (w_n) \text{ is not bounded below}$$

Therefore, $\lim\sup(w_n) = -\infty$

$\lim\inf(w_n) = -\infty$.

Remark. Every bounded sequence has limit superior and limit inferior, which are unique, finite real numbers. Even unbounded sequences have limit superior and limit inferior. If a sequence is not bounded above, its limit superior is $+\infty$, and if a sequence is not bounded below, its limit inferior is $-\infty$.

In the following result, we state some elementary properties of limit superiors and limit inferiors of sequences.

Theorem 2.8.4. Let (x_n) be any sequence of real numbers. We have,

(a) $\lim\inf(x_n) \le \lim\sup(x_n)$;

(b) If (x_n) is bounded above by K then $\lim\sup(x_n) \le K$;

(c) If (x_n) is bounded below by k, then $k \le \lim\inf(x_n)$;

(d) If (x_{n_k}) is any subsequence of (x_n), then

$$\lim\inf(x_n) \le \lim\inf(x_{n_k}) \le \lim\sup(x_{n_k}) \le \lim\sup(x_n).$$

Proof. (a) For each $n \in \mathbf{N}$

$$\underline{x_n} = \inf\{x_n, x_{n+1}, \ldots\}$$

$$\le \sup\{x_n, x_{n+1}, \ldots\}$$

$$= \bar{x}_n$$

Hence, $\lim(\underline{x_n}) \le \lim(\bar{x}_n)$. That is, $\liminf(x_n) \le \limsup(x_n)$

(b) Follows from the fact that

$$\bar{x}_n \le \ldots \le \bar{x}_2 \le \bar{x}_1 = \sup\{x_1, x_2, \ldots, x_n, \ldots\}$$
$$\le K$$

(c) Follows from the fact that

$$k \le \inf\{x_1, x_2, \ldots, x_n, \ldots\}$$
$$= \underline{x_1}$$
$$\le \underline{x_2} \le \underline{x_3} \le \ldots \le \underline{x_n}.$$

(d) For any k ∈ **N**, we have, $n_k \ge k$. Therefore, $\{x_{n_k}, x_{n_{k+1}}, \ldots\} \subseteq \{x_k, x_{k+1}, \ldots\}$.

Hence, $\sup\{x_{n_k}, x_{n_{k+1}}, \ldots\} \le \sup\{x_k, x_{k+1}, \ldots\}$

and $\quad \inf\{x_{n_k}, x_{n_{k+1}}, \ldots\} \ge \inf\{x_k, x_{k+1}, \ldots\}$

That is $\bar{x}_{n_k} \le \bar{x}_k$ and $\underline{x_{n_k}} \ge \underline{x_k}$

Hence, $\lim(\overline{x_{n_k}}) \le \lim(\overline{x_k})$ and $\lim(\underline{x_{n_k}}) \ge \lim(\underline{x_k})$.

Thus, $\quad \limsup(x_{n_k}) \le \limsup(x_k)$ and $\liminf(x_{n_k}) \ge \liminf(x_k)$.

Therefore, we have

$$\liminf(x_n) \le \liminf(x_{n_k}) \le \limsup(x_{n_k}) \le \limsup(x_n).$$

Theorem (Criterion for limit superior) 2.8.5. Let (x_n) be a bounded sequence and L be a real number. Then, L = lim sup (x_n) if and only if the following two conditions hold

(i) For each ε > 0, $x_n < L + \varepsilon$ for all except finitely many values of n

(ii) For each ε > 0, $x_n > L - \varepsilon$ for infinitely many values of n.

Proof. Let L = lim sup (x_n). Then, $L = \lim(\bar{x}_n)$, where $\bar{x}_n = \sup\{x_n, x_{n+1}, \ldots\}$.

Hence, for each ε > 0, there exists some m ∈ **N** such that

$$|\bar{x}_n - L| < \varepsilon \qquad \text{for all } n \ge m$$

Then, $\quad \bar{x}_n < L + \varepsilon \qquad$ for all $n \ge m$

$\Rightarrow \quad \sup\{x_n, x_{n+1}, x_{n+2}, \ldots\} < L + \varepsilon$ for all $n \ge m$

$\Rightarrow \quad x_n < L + \varepsilon \qquad$ for all $n \ge m$

Then condition (i) is satisfied

To prove condition (ii), we observe that

$$\bar{x}_1 \geq \bar{x}_2 \geq \bar{x}_3 \geq \ldots \geq \bar{x}_n \geq \ldots$$

That is $(\bar{x}_n)$ is a decreasing sequence.

Hence, $L = \lim(\bar{x}_n) = \inf\{\bar{x}_n : n \in \mathbf{N}\}$, by the Monotone Convergence Theorem.

Then, $L \leq \bar{x}_k$ for each $k \in \mathbf{N}$

Hence, for each $\varepsilon > 0$,

$$L - \varepsilon < \bar{x}_k \quad \text{for each } k \in \mathbf{N}.$$

But $\bar{x}_k = \sup\{x_k, x_{k+1}, \ldots\}$.

Hence, by property of supremum, there exists some member in $\{x_k, x_{k+1}, \ldots\}$, say, x_{n_k} with $n_k \geq k$ such that $x_{n_k} \geq L - \varepsilon$.

This holds for each $k \in \mathbf{N}$ and in particular for $n_k + 1$. Thus, there exists, say, $x_{n_{k+1}}$, with $n_{k+1} \geq n_k + 1 > n_k$. Hence, there are infinitely many values of n for which $x_n > L - \varepsilon$.

Conversely, let conditions (i) and (ii) hold.

Condition (i) implies that for a give $\varepsilon > 0$, there exists $m \in \mathbf{N}$ such that

$$x_n < L + \varepsilon \quad \text{for all } n \geq m.$$

$$\Rightarrow \quad \bar{x}_n \leq L + \varepsilon \quad \text{for all } n \geq m$$

$$\Rightarrow \quad \lim(\bar{x}_n) \leq L + \varepsilon$$

Since, ε is arbitrary, this implies that $\lim(\bar{x}_n) \leq L$.

Again, by condition (ii), $\bar{x}_n = \sup\{x_k : k \geq n\} > L - \varepsilon$.

Hence, $\lim(\bar{x}_n) \geq L - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $\lim(\bar{x}_n) \geq L$.

Consequently, $\lim(\bar{x}_n) = L$. That is $L = \lim\sup(x_n)$.

Similar condition for limit inferior are stated below.

Theorem (Criterion for limit inferior) 2.8.6. Let (x_n) be a bounded sequence and L be any real number. Then L = limit inf (x_n) if and only if the following conditions hold :

(i) For each $\varepsilon > 0$, $x_n > L - \varepsilon$ for all except finitely many values of n

(ii) For each $\varepsilon > 0$, $x_n < L + \varepsilon$ for infinitely many values of n.

The next theorem provides a necessary as well as sufficient condition for the convergence of a bounded sequence in terms of limit superior and limit inferior.

Theorem 2.8.7. A bounded sequence (x_n) converges to a real number x if and only if

$$\lim\sup(x_n) = \lim\inf(x_n) = x.$$

Proof. Let (x_n) be a bounded sequence which converges to x.

Then, for $\varepsilon > 0$, there exists $m \in \mathbf{N}$ such that

$$x - \varepsilon < x_n < x + \varepsilon \qquad \text{for all } n \geq m.$$

Then, by Theorem 2.8.5 and Theorem 2.8.6,

$$\lim\sup (x_n) = x \quad \text{and} \quad \lim\inf (x_n) = x \text{ respectively.}$$

Conversely, let $\lim\sup (x_n) = x$. Then, by Theorem 2.8.5 for $\varepsilon > 0$, there exists $m_1 \in \mathbf{N}$ such that $x_n < x + \varepsilon$ for all $n \geq m_1$.

Similarly, by Theorem 2.8.6, there exists $m_2 \in \mathbf{N}$ such that $x - \varepsilon < x_n$ for all $n \geq m_2$.

Hence, for all $n \geq m = \max \{m_1, m_2\}$, we get $x - \varepsilon < x_n < x + \varepsilon$ for all $n \geq m$.

That is $|x_n - x| < \varepsilon$ for all $n \geq m$

Hence, (x_n) converges to x.

Illustrative Examples

1. Find out the limit superior and limit inferior of the following seqeunces

(a) $(1+(-1)^n)$ (b) $\left(\frac{1}{n}\right)$

(c) $(-n^2)$

Solution. (a) Here $x_n = 1+(-1)^n = \begin{cases} 0, & \text{when n is odd} \\ 2, & \text{when n is even} \end{cases}$

Hence, $(x_n) = (0,2,0,2,0,2,0,2...)$

$$\overline{x}_n = 2, \quad \underline{x}_n = 0 \qquad \text{for each } n \in \mathbf{N}$$

Hence, $\lim\sup(x_n) = \lim(\overline{x}_n) = 2$

$$\lim\inf (x_n) = \lim(\underline{x}_n) = 0$$

(b) $\left(\frac{1}{n}\right) = \left(1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...\right)$

$$\overline{x}_n = \sup\left\{\frac{1}{n}, \frac{1}{n+1}, ...\right\} = \frac{1}{n}$$

$$\underline{x} = \inf\left\{\frac{1}{n}, \frac{1}{n+1}, ...\right\} = 0$$

Therefore,

$$\lim\sup(x_n) = \lim(\overline{x}_n) = \lim\left(\frac{1}{n}\right) = 0$$

$$\lim\inf (x_n) = \lim(\underline{x}_n) = 0$$

(c) $(-n^2) = (-1^2, -2^2, -3^2, -4^2, ...)$

$$\overline{x}_n = \sup\{-n^2, -(n+1)^2, ...\} = -n^2$$

$$\underline{x}_n = \inf\{-n^2, -(n+1)^2, ...\} = -\infty, \quad \text{as } (-n^2) \text{ is not bounded below}$$

Hence,

$$\lim\sup(x_n) = \lim(\overline{x}_n) = \lim(-n^2) = -\infty$$

$$\lim\inf(x_n) = -\infty$$

2. Find out the limit superior and limit inferior of the following sequences

(a) $(s_n) = (0,1,2,1,0,1,2,1,0,1,2,1,0,1,2,1,0,...)$

(b) $(t_n) = (2,1,1,0,2,1,1,0,2,1,1,0,...)$

Also find $\lim\sup(s_n + t_n)$, $\lim\inf(s_n + t_n)$, $\lim\sup(s_n t_n)$, $\lim\inf(s_n t_n)$

Here $\overline{s}_n = 2,\ \underline{s}_n = 0$

$\overline{t}_n = 2,\ \underline{t}_n = 0$

Hence, $\lim\sup(s_n) = \lim(\overline{s}_n) = 2$

$$\lim\inf(s_n) = \lim(\underline{s}_n) = 0$$

$$\lim\sup(t_n) = \lim(\overline{t}_n) = 2$$

$$\lim\inf(t_n) = \lim(\underline{t}_n) = 0$$

$$(s_n + t_n) = (0+2, 1+1, 2+1, 1+0, 0+2, 1+1, ...)$$

$$= (2,2,3,1,2,2,...)$$

$$s_n t_n = (0.2, 1.1, 2.1, 1.0, 0.2, 1.1, ...)$$

$$= (0,1,2,0,0,1,...)$$

$$\overline{s_n + t_n} = 3, \quad \underline{s_n + t_n} = 1$$

$$\overline{s_n t_n} = 2, \quad \underline{s_n t_n} = 0$$

Hence, $\lim\sup(s_n + t_n) = \lim(\overline{s_n + t_n}) = 3$

$$\lim\inf(s_n + t_n) = \lim(\underline{s_n + t_n}) = 1$$

$$\lim\sup(s_n t_n) = \lim(\overline{s_n t_n}) = 2$$

$$\lim\inf(s_n t_n) = \lim(\underline{s_n t_n}) = 0$$

EXERCISE 2.8

1. Find out the limit superior and limit inferior of the following sequences

(a) $\left((-1)^n\left(1+\frac{1}{n}\right)\right)$ (b) $\left(n(1+(-1)^n\right)$

(c) $\left(1+\frac{(-1)^n}{n}\right)$ (d) $\left(\left(1+\frac{1}{n}\right)^{n+1}\right)$

(e) $\left(\frac{(-1)^n}{n^2}\right)$ (f) $\left(\frac{1}{2^n}\right)$

(g) $\left((-2)^n\left(1+\frac{1}{n}\right)\right)$ (h) $\left((-1)^n\frac{1}{n}\right)$

Verify the following properties of limit superior and and Σa_n converges ?