Linear homogenous equations with constant coefficients

Consider the n^{th} order homogenous linear equation in which all of the coefficients are real constants

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1} \frac{dy}{dx} + a_n y = 0 \quad \text{---} (1)$$

where $a_0, a_1, a_2, \dots, a_{n-1}, a_n$ are real constants.

To find the general solution, we first look for a single solution of (1). We begin by assuming a solution of the form $y = e^{mx}$. Substituting this value of y & its derivatives in (1), we get

$$a_0 m^n e^{mx} + a_1 m^{n-1} e^{mx} + \dots + a_{n-1} m e^{mx} + a_n e^{mx} = 0$$

$$\Rightarrow e^{mx} \left[a_0 m^n + a_1 m^{n-1} + \dots + a_{n-1} m + a_n \right] = 0$$

Since e^{mx} is never zero, so $y = e^{mx}$ will be a solution of (1) if m is a root of the equation

$$a_0 m^n + a_1 m^{n-1} + \dots + a_{n-1} m + a_n = 0 \quad \text{---} (2)$$

Equation (2) is called the characteristic equation or auxillary equation of differential equation (1).

Theorem 1 (Only statement) (a) If auxillary equation (2) of linear differential equation (1) has n distinct real roots $m_1, m_2, \dots, m_n$ then general solution of (1) is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$ where $c_1, c_2, \dots, c_n$ are arbitrary constants.

(b) If (2) has repeated roots m occuring k times the part of general solution ① corresponding to this k-fold repeated root is

$$(c_1 + c_2 x + \dots + c_k x^{k-1}) e^{mx}.$$

and if remaining roots are distinct real numbers $m_{k+1}, \dots, m_n$ then general solution of ① is

$$y = (c_1 + c_2 x + \dots + c_k x^{k-1}) e^{mx} + c_{k+1} e^{m_{k+1} x} + \dots + c_n e^{m_n x}$$

Theorem 2 (Only Statement) (a) If auxillary equation ② has conjugate complex roots $\alpha + i\beta$ and $\alpha - i\beta$ neither repeated, then corresponding part of general solution of ① may be written as

$$y = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x).$$

(b) If however, $\alpha + i\beta$ and $\alpha - i\beta$ are each k-fold roots of auxillary equation, then corresponding part of general solution of ① is

$$y = e^{\alpha x} \left[(c_1 + c_2 x + \dots + c_k x^{k-1}) \cos \beta x + (c_1' + c_2' x + \dots + c_k' x^{k-1}) \sin \beta x \right].$$

Example : Find a general solution of

$$\frac{d^2 y}{dx^2} - 5 \frac{dy}{dx} + 6y = 0$$

Auxillary equation is $m^2 - 5m + 6 = 0$.

It has 2 distinct roots $m = 2, 3$

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The general solution is $y = c_1 e^{2x} + c_2 e^{3x}$.

Example: Find a general solution of $y'' - 8y' + 16y = 0$

Auxillary equation is $m^2 - 8m + 16 = 0$

It has repeated root with multiplicity 2 $m = 4, 4$

The general solution is $y = (c_1 + c_2 x) e^{4x}$.

Example: Find a general solution of $y'' - 4y' + 13y = 0$

Auxillary equation is $m^2 - 4m + 13 = 0$

It has complex roots $m = \frac{4 \pm \sqrt{16 - 4(13)}}{2}$

$$= \frac{4 \pm i6}{2} = 2 \pm i3$$

General solution is $y = e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$.

Q solve $\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 7\frac{dy}{dx} - 3y = 0$

Auxillary equation is $m^3 - 5m^2 + 7m - 3 = 0$

Clearly, $m = 1$ is a solution of auxillary equation

$$
\begin{array}{l}
m-1 \,\big)\, m^3 - 5m^2 + 7m - 3 \,\big(\, m^2 - 4m + 3 \\
\quad\; m^3 - m^2 \\
\quad\; - \quad + \\
\hline
\quad\; -4m^2 + 7m - 3 \\
\quad\; -4m^2 + 4m \\
\quad\;\; + \quad - \\
\hline
\qquad\quad 3m - 3 \\
\qquad\quad 3m - 3 \\
\hline
\qquad\quad \times
\end{array}
$$

$m^3 - 5m^2 + 7m - 3 = 0$

$\therefore \Rightarrow (m-1)(m^2 - 4m + 3) = 0$

$\Rightarrow m = 1, 1, 3$

General solution is

$y = (c_1 + c_2 x)e^{x} + c_3 e^{3x}$.

Q Solve $\frac{d^3y}{dx^3} + 8y = 0$

Auxillary equation is $m^3 + 8 = 0$

$\Rightarrow m^3 + 2^3 = 0$

$\Rightarrow (m+2)(m^2 - 2m + 4) = 0$

Roots are $m = -2,\ 1 \pm i\sqrt{3}$

General solution is $y = c_1 e^{-2x} + e^{x}\left[c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x\right]$.

Non-homogenous linear equation with constant coefficients

Consider the n^{th} order (non-homogenous) linear differential equation

$$a_0 \frac{d^ny}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \dots + a_{n-1}\frac{dy}{dx} + a_n y = V(x) \quad \text{— (A)}$$

and the corresponding homogenous equation

$$a_0 \frac{d^ny}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \dots + a_{n-1}\frac{dy}{dx} + a_n y = 0 \quad \text{— (B)}$$

1. The general solution of (B) is called complementary function (CF) of equation (A).
2. Any particular solution of (A) involving no arbitrary constants is called a particular solution of (A) (PI)
3. The solution $y = CF + PI$ of (A) where CF is the complementary function & PI is the particular integral of (A) is called the general solution of (A).

Let $D \equiv \frac{d}{dx}$

Then (B) can be written as $\left[a_0 D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n\right] y = 0$

or $f(D)\, y = 0$

The solⁿ of ~~algeb~~ LDE consists of 2 parts

$$a_0\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+\dots+a_{n-1}\frac{dy}{dx}+a_ny = V \quad (2)$$ where V is a funⁿ of x.

(i) Complimentary soln

(ii) Particular Integral.

The gen solⁿ is $y = CF + PI$.

CF: CF of (1) is gen solⁿ of diff eqⁿ $f(D)y = 0$.

P.I: Now (2) can be written as $f(D)y = V \Rightarrow y = \frac{1}{f(D)}V$.

$PI = \frac{1}{f(D)}V$ (To be evaluated acc. to rules)

Rules:

(1) $\boxed{\text{When } V = k \text{ (const) then } \frac{1}{f(D)}k = \frac{k}{\text{const term of } f(D)}}$

eg: $\frac{1}{D^2+2D+3}.5 = \frac{5}{3}$.

(2) $\boxed{\text{If } V = e^{mx} \text{ then } \frac{1}{f(D)}e^{mx} = \frac{1}{f(m)}e^{mx}}$ $\boxed{\text{Replace } D\to m \text{ provided } f(m)\neq 0}$

eg: $\frac{1}{D^2+2D+3}e^{-2x} = \frac{1}{(-2)^2+2(-2)+3}e^{-2x} = \frac{e^{-2x}}{3}$

$\boxed{\text{If } f(m)=0 \text{ then } \frac{1}{f(D)}e^{mx} = x\frac{1}{f'(D)}e^{mx}}$

eg: $\frac{1}{D^2-3D+2}e^{2x}$ Then $f(2) = 4-6+2=0$ $\therefore \frac{1}{f(D)}e^{2x} = x.\frac{1}{(2D-3)}e^{2x}$

$= x\frac{1}{[2(2)-3]}e^{2x} = xe^{2x}$

(3) $\boxed{V = \cos mx \text{ or } \sin mx}$

Subcase: when $f(D)$ contains even powers of D $f(D) = \phi(D^2)$

$\frac{1}{\phi(D^2)}\cos mx = \frac{1}{\phi(-m^2)}\cos mx$ $\boxed{\text{provided } \phi(-m^2)\neq 0 \mid D^2 \to -m^2}$

$\frac{1}{\phi(D^2)}\sin mx = \frac{1}{\phi(-m^2)}\sin mx$

eg $\frac{1}{D^2+1}\sin 2x = \frac{1}{-(2)^2+1}\sin 2x = -\frac{1}{3}\sin 2x$.

If $\phi(-m^2)=0$ Then $\frac{1}{\phi(D^2)}V = x\frac{1}{\phi'(D)}V$ where $V = \cos mx$ or $\sin mx$

eg: $\frac{1}{D^2+1}\sin x$ $\phi(D^2) = D^2+1$, $\phi(-m^2) = -1+1 = 0$ then $\frac{1}{D^2+1}\sin x = x.\frac{1}{2D}\sin x$

$= -\frac{x\cos x}{2}$.

$\boxed{D = \frac{d}{dx}, \quad \frac{1}{D} = \int, \quad D.\frac{1}{D} = 1}$

General procedure :—

for $\frac{1}{f(D)}\cos mx$ or $\frac{1}{f(D)}\sin mx$:—

Step 1 Replace $D^2 \to -m^2$ & do not change D. Eqⁿ reduces to $\frac{1}{aD+b}$ ~~cos mx~~ or $\frac{1}{aD+b}\sin mx$

step 2: $\frac{1}{aD+b}\cos mx = \frac{aD-b}{(aD+b)(aD-b)}\cos mx = \frac{aD-b}{a^2D^2-b^2}\cos mx$

Now again replace $D^2 \to -m^2$

$= \frac{aD-b}{-a^2m^2-b^2}\cos mx = \frac{a}{-a^2m^2-b^2}\left[D\cos mx - b\cos mx\right]$

$= \frac{-a}{a^2m^2+b^2}\left[-m\sin mx - b\cos mx\right].$

eg: $\frac{1}{D^3+1}\cos x = \frac{1}{D(D^2)+1}\cos x$ $\quad D^2 \to -1^2 = -1$

$= \frac{1}{-D+1}\cos x = \frac{1+D}{(1-D)(1+D)}\cos x = \frac{1+D}{1-D^2}\cos x = \frac{1+D}{1-(-1)}\cos x$

$= \left(\frac{1+D}{2}\right)\cos x = \frac{1}{2}\left[\cos x + D\cos x\right] = \frac{1}{2}\cos x + \frac{\sin x}{2}.$

Rule 4: $V = x^m$. $\quad$ use the foll^n :-

$(1-a)^{-1} = 1+a+a^2+a^3+ \cdots$

$(1+a)^{-1} = 1-a+a^2-a^3+ \cdots$

eg:- $\frac{1}{D^2-3D-4}x^2$

$= \frac{1}{-4\left[1-\frac{D^2-3D}{4}\right]}x^2 = \frac{-1}{4}\left[1-\left(\frac{D^2-3D}{4}\right)\right]^{-1}x^2$

$= -\frac{1}{4}\left[1+\left(\frac{D^2-3D}{4}\right)+\left(\frac{D^2-3D}{4}\right)^2+ \cdots\right]x^2$

$= -\frac{1}{4}\left[1+\frac{D^2}{4}-\frac{3D}{4}+\frac{D^4}{16}+\frac{9D^2}{16}-\frac{3D^3}{8}+ \cdots\right]x^2$ $\quad Dx^2 = 2x, \; D^2x^2 = 2$

$= -\frac{1}{4}\left[x+\frac{D^2x^2}{4}-\frac{3}{4}Dx+\frac{D^4x}{16}+\frac{9}{16}D^2x-\frac{3}{8}D^3x^2+ \cdots\right]$

$= -\frac{1}{4}\left\{x+\frac{2}{4}-\frac{3}{4}(2x)+0+\frac{9}{16}(2)-0 \cancel{+ \cdots}\right\}$ as remaining terms vanish.

Q Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{4x}$ $\quad$ sol^n $y = c_1e^{3x}+c_2e^{2x}+\frac{1}{2}e^{4x}$.

Q $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 2e^{2x}$. $\quad$ Sol^n $y = e^{-x}\left[c_1+c_2x\right]+\frac{2}{9}e^{2x}$.

Q $\frac{d^3y}{dx^3} - y = (e^x+1)^2$ $\quad$ sol^n $y = e^{-\frac{1}{2}x}\left[c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right] + e^x\left(c_3+\frac{2}{3}x\right)+\frac{1}{7}e^{2x}-1$.

Q $\frac{d^3y}{dx^3} + 8y = x^4+2x+1$

Sol:- $y = c_1e^{-2x}+e^x(c_2\cos\sqrt{3}x + c_3\sin\sqrt{3}x) + \frac{1}{8}(x^4-x+1)$.

Q $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - \frac{dy}{dx} - y = \cos 2x$

Sol^n: $y = c_1e^x + (c_2+c_3x)e^{-x} - \frac{2}{25}\sin 2x - \frac{\cos 2x}{25}$

compound form $e^{ax} V = RHS$.

Then $\boxed{\dfrac{1}{f(D)} e^{ax} V = e^{ax} \dfrac{1}{f(D+a)} V}$

Q $\dfrac{d^2y}{dx^2} + y = xe^{2x}$

$\dfrac{1}{D^2+1} e^{2x} \cdot x = e^{2x} \dfrac{1}{(D+2)^2+1} x = e^{2x} \dfrac{1}{(D^2+4+4D)+1} \cdot x = \dfrac{e^{2x}}{5} \dfrac{1}{\left(1+\frac{D^2+4D}{5}\right)} x$

$= \dfrac{e^{2x}}{5} \left[\left(1+\left(\dfrac{D^2+4D}{5}\right)\right)\right]^{-1} x = \dfrac{e^{2x}}{5} \left[1 - \dfrac{D^2+4D}{5}\right] x = \dfrac{e^{2x}}{5}\left[x - \dfrac{4}{5}\right]$

complete sol^n- $y = c_1 \cos x + c_2 \sin x + \dfrac{e^{2x}}{25}(5x-4)$.

Q $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = e^{2x} \sin x$.

CF: $D^2+3D+2=0$. $D = -2, -1$ $CF = c_1 e^{-2x} + c_2 e^{-x}$,

PI $= \dfrac{1}{D^2+3D+2} (e^{2x} \sin x) = e^{2x} \dfrac{1}{(D+2)^2+3(D+2)+2} (\sin x)$

$= e^{2x} \dfrac{1}{D^2+7D+12} \sin x$ $m=1$, $-m^2=-1$, $D^2 \to -m^2$

$= e^{2x} \dfrac{1}{-1+7D+12} \sin x = e^{2x} \dfrac{1}{7D+11} \sin x = e^{2x} \dfrac{1}{7D+11} \times \dfrac{7D-11}{7D-11} \sin x$

$= e^{2x} \dfrac{7D-11}{49D^2-121} \sin x = \dfrac{e^{2x}}{-170}(7D-11)\sin x$ $\therefore y = c_1 e^{-2x} + c_2 e^{-x} + \dfrac{e^{2x}}{170}(11\sin x - 7\cos x)$

Q $\left(\dfrac{d^3y}{dx^3} - 2\dfrac{dy}{dx} + 4y\right) = e^x \cos x$.

CF = $D=-2$, $1 \pm i$ (By trial) $CF = c_1 e^{-2x} + e^x(c_2 \cos x + c_3 \sin x)$

PI $\dfrac{1}{D^3-2D+4}(e^x \cos x) = e^x \dfrac{1}{(D+1)^3-2(D+1)+4} (\cos x) = e^x \dfrac{1}{D^3+3D^2+D+3} \cos x$

Now $m=1$ $-m^2=-1$ $f(-m^2)=0$

$\therefore = e^x \cdot x \dfrac{1}{3D^2+6D+1} \cos x$. $D^2 \to -m^2 = -1$.

$y = c_1 e^{-2x} + e^x(c_2 \cos x + c_3 \sin x) + \dfrac{xe^x}{20}(3\sin x - \cos x)$

Q $\dfrac{d^2y}{dx^2} + 2y = x^2 e^{3x} + e^x \cos 2x$

$m = \pm\sqrt{2}\, i$ CF $= c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x$

PI $= \underbrace{\dfrac{1}{D^2+2} x^2 e^{3x}}_{①} + \underbrace{\dfrac{1}{D^2+2} e^x \cos 2x}_{②} =$ ① $= e^{3x} \dfrac{1}{(D+3)^2+2} x^2$

② $= e^x \dfrac{1}{(D+1)^2+2} \cos 2x$

$y = c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x + \dfrac{e^{3x}}{11}\left(x^2 - \dfrac{12}{11}x + \dfrac{50}{121}\right) - \dfrac{1}{17} e^x(\cos 2x - 4\sin 2x)$

RHS = $x^m \sin ax$ or $x^m \cos ax$

To introduce e^{iax} :- $e^{i\theta} = \cos\theta + i\sin\theta$

$x^m \sin ax$ = Imag. part of $e^{iax} x^m$

$x^m \cos ax$ Real part of $e^{iax} x^m$

Q $\frac{d^2y}{dx^2} + 4y = x\sin x$

$D = \pm 2i$ CF = $c_1 \cos 2x + c_2 \sin 2x$

PI. = $\frac{1}{D^2+4} x\sin x = \text{Im}\left\{\frac{1}{D^2+4} e^{ix}.x\right\}$

Now $\frac{1}{D^2+4} e^{ix}.x = e^{ix}\frac{1}{(D+i)^2+4}x = e^{ix}\frac{1}{D^2-1+2iD+4}x = e^{ix}\frac{1}{D^2+2iD+3}x$

$= \frac{e^{ix}}{3}\frac{1}{\left[1+\frac{D^2+2iD}{3}\right]}x = \frac{e^{ix}}{3}\left(1+\left(\frac{D^2+2iD}{3}\right)\right)^{-1}x = \frac{e^{ix}}{3}\left[1-\frac{2i}{3}D\right]x$

$= \frac{e^{ix}}{3}\left[x - \frac{2i}{3}\right] = \frac{(\cos x + i\sin x)}{3}\left(x - \frac{2i}{3}\right)$

Now $\text{Im}\left\{\frac{1}{D^2+4}e^{ix}.x\right\} = \frac{x\sin x}{3} - \frac{2}{9}\cos x$

$\therefore y = CF + PI = c_1\cos 2x + c_2 \sin 2x + \frac{x\sin x}{3} - \frac{2}{9}\cos x$.

Q $\frac{d^2y}{dx^2} - y = x^2\cos x$. $m = \pm 1$

CF = $c_1 e^x + c_2 e^{-x}$.

PI = Real $\left\{\frac{1}{D^2-1} x^2.e^{ix}\right\}$ = Real $\left\{e^{ix}\frac{1}{(D+i)^2-1}x^2\right\}$

$= -\frac{1}{2}\left[(x^2-1)\cos x - 2x\sin x\right]$. $y = CF + PI$

Q $\left(\frac{d^2y}{dx^2} + y\right) = x^2\sin 2x$ PI = Im $\left\{\frac{1}{D^2+1}x^2e^{2ix}\right\}$

$D = \pm i$

CF = $c_1\cos x + c_2 \sin x$ $= -\frac{8}{9}x\cos 2x - \frac{\sin 2x}{3}.x^2 + \frac{26}{27}\sin 2x$

$y = CF + PI$

Cauchy-Euler Equation

$$\alpha_0 x^3 \frac{d^3y}{dx^3} + \alpha_1 x^2 \frac{d^2y}{dx^2} + \alpha_2 x \frac{dy}{dx} + \alpha_3 y = \phi(x) \quad \text{---(1)}$$

Method :- let $z = \log x$ ie $x = e^z$.

$\frac{dy}{dz} = \frac{dy}{dx} / \frac{dx}{dz}$

$D \equiv \frac{d}{dz}$ $\quad x\frac{dy}{dx} = Dy$

$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$

$\Rightarrow x\frac{dy}{dx} = Dy$

||ly $x^2 \frac{d^2y}{dx^2} = D(D-1)y$.

$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$.

Then (1) reduces to a LDE with constt coeff in y & z.
Solve as in prev. chapter & then replace z by log x.

Q $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = (1-x)^2$ —(1)

$= 1 + x^2 - 2x$.

let $z = \log x$ $\quad = 1 + e^{2z} - 2e^z$,

Then (1) becomes $[D(D-1) + 3D + 1]y =$

CF $= (c_1 + c_2 z)e^{-z} = (c_1 + c_2\log x)x^{-1}$

CF : $D^2 + 2D + 1 = 0 \Rightarrow D = -1, -1$

PI $= \frac{1}{D^2+2D+1}[1 + e^{2z} - 2e^z]$

(i) $= \frac{1}{1} = 1$ $\quad$ (ii) $\frac{1}{2^2+2(2)+1}e^{2z} = \frac{1}{9}e^{2z} = \frac{1}{9}x^2$ $\quad$ (iii) $\frac{-2}{1+2+1}e^z = -\frac{1}{2}x$.

$y = (c_1 + c_2\log x)x^{-1} + 1 + \frac{1}{9}x^2 - \frac{1}{2}x$.

Q $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x + x^3 + x^2\log x$ $\quad D \equiv \frac{d}{dz}$

let $z = \log x$ ie $x = e^z$

$[D(D-1) - 2D + 2]y = e^z + e^{3z} + e^{2z}\cdot z$

CF $= c_1e^{2z} + c_2e^z = c_1x^2 + c_2x$.

CF : $D^2 - 3D + 2 = 0$ $\quad D = 2, 1$

PI : $\frac{1}{(D^2-3D+2)}[e^z + e^{3z} + e^{2z}\cdot z]$

(i) $-ze^z = -\log x\cdot x$

(ii) $\frac{1}{2}e^{3z} = \frac{x^3}{2}$

(iii) $\frac{1}{D^2-3D+2}e^{2z}\cdot z = e^{2z}\frac{1}{(D+2)^2-3(D+2)+2}\cdot z$ (First Integrate)

$= e^{2z}\frac{1}{D^2+D}z = e^{2z}\frac{1}{D+1}\cdot\frac{1}{D}z = e^{2z}\frac{1}{D+1}\frac{z^2}{2}$

$= \frac{e^{2z}}{2}(1+D)^{-1}z^2 = \frac{e^{2z}}{2}(1 - D + D^2)z^2 = \frac{e^{2z}}{2}(z^2 - 2z + 2) = \frac{x^2}{2}[(\log x)^2 - 2\log x + 2]$

$y = CF + PI = c_1x^2 + c_2x - x\log x + \frac{x^3}{2} + \frac{x^2}{2}(\log x)^2 - x^2\log x$ (x^2 already included in CF)

Q $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$

$D = -1, 1\pm i$ $\quad y = c_1x^{-1} + x[c_2\cos(\log x) + c_3\sin(\log x)] + 5x + 2\log x\cdot x^{-1}$

Q $x\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} = \frac{1}{x}$ (Multiply by x^2). $\quad D = 0, 1, 1$

$y = c_1 + [c_2 + c_3\log x]x + \frac{(\log x)^2}{2}\cdot x$

Form:— Reducible to Euler's eqⁿ

$$\alpha_0(ax+b)^2\frac{d^2y}{dx^2} + \alpha_1(ax+b)\frac{dy}{dx} + \alpha_2 y = \phi(x) \quad \text{—(1)}$$

let $v = ax+b$

$$\frac{dy}{dx} = \frac{dy}{dv}\cdot\frac{dv}{dx} = a\cdot\frac{dy}{dv}$$

$$\frac{d^2y}{dx^2} = a\frac{d^2y}{dv^2}\cdot\frac{dv}{dx} = a^2\frac{d^2y}{dv^2}$$

Thus (1) becomes,

$$\alpha_0 a^2 v^2\frac{d^2y}{dv^2} + \alpha_1 a v\frac{dy}{dv} + \alpha_2 y = \phi\left(\frac{v-b}{a}\right)$$

Euler eqⁿ in y (dep) & v (indep).

Q. $(3x+2)^2\dfrac{d^2y}{dx^2} + 3(3x+2)\dfrac{dy}{dx} - 36y = 3x^2+4x+1$ —(1)

let $v = 3x+2$

$\therefore \dfrac{dy}{dx} = 3\dfrac{dy}{dv}$ & $\dfrac{d^2y}{dx^2} = 9\dfrac{d^2y}{dv^2}$

$\therefore$ (1) becomes $9v^2\dfrac{d^2y}{dv^2} + 3\cdot(3v)\dfrac{dy}{dv} - 36y = 3\left(\dfrac{v-2}{3}\right)^2 + 4\left(\dfrac{v-2}{3}\right)+1$

$\Rightarrow v^2\dfrac{d^2y}{dv^2} + v\dfrac{dy}{dv} - 4y = \dfrac{1}{27}(v^2-1)$. (Cauchy euler form) in y & v.

let $z = \log v$ $\quad D = \dfrac{d}{dz}$

$$[D(D-1)+D-4]y = \frac{1}{27}(e^{2z}-1)$$

CF : $D = \pm 2$ $\quad$ CF : $c_1e^{2z}+c_2e^{-2z} = c_1v^2+c_2v^{-2} = c_1(3x+2)+c_2(3x+2)^{-2}$

PI $\dfrac{1}{D^2-4}\left[\dfrac{e^{2z}}{27} - \dfrac{1}{27}\right]$ $\quad$ (i) $\dfrac{1}{f(2)} = 0$ $\quad\therefore (PI)_1 = \dfrac{1}{27}\cdot z\cdot\dfrac{1}{2D}e^{2z} = \dfrac{z}{54}\dfrac{e^{2z}}{2}$

$$= \frac{\log v}{108}\cdot v^2 = \frac{\log(3x+2)\cdot(3x+2)^2}{108}$$

(ii) $\dfrac{1}{D^2-4}\left(\dfrac{-1}{27}\right) = \dfrac{-1/27}{-4} = \dfrac{1}{108}$ $\quad$ $y = CF + PI$.

Q $(5+2x)^2\dfrac{d^2y}{dx^2} - 6(5+2x)\dfrac{dy}{dx} + 8y = 0$

$$y = c_1(5+2x)^{2+\sqrt{2}} + c_2(5+2x)^{2-\sqrt{2}}.$$

Q $(x+1)^2\dfrac{d^2y}{dx^2} - 3(x+1)\dfrac{dy}{dx} + 4y = x^2$

$$y = [c_1+c_2\log(x+1)](x+1)^2 + \frac{1}{4} - 2(x+1) + \frac{1}{2}(x+1)^2\{\log(x+1)\}^2.$$

Method of Variation of parameters.

Diff eq^n of type $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ (const coeff or fun^n of x)

Step ①:- $CF = c_1u + c_2v$

② Wronskian $W(u,v) = uv' - u'v$

③ $f(x) = -\frac{vR}{W}$ & $g(x) = \frac{uR}{W}$

④ $F(x) = \int f(x)dx$ & $G(x) = \int g(x)dx$ (without adding consts.)

⑤ $PI = uF + vG$ ⑥ $y = CF + PI$

Note: Use when RHS is not in std form $\log x$ etc. or if specifically asked.

eg: RHS $= \sec n\theta$, $\tan n\theta$, $\frac{1}{1+e^x}$,

Q $\frac{d^2y}{dx^2} + y = \tan x$ $P=0$ $Q=1$. $R=\tan x$ $u=\cos x$ $v=\sin x$

CF: $D^2+1=0$ $D=\pm i$ CF: $c_1\cos x + c_2\sin x$

$W = uv' - u'v = \cos^2 x + \sin^2 x = 1$.

$f(x) = -\frac{vR}{W} = -\sin x . \tan x = -\frac{\sin^2 x}{\cos x}$ $g(x) = \cos x \tan x = \sin x$.

$F(x) = \int f(x)dx = \int \frac{-\sin^2 x}{\cos x} dx = \int \frac{-1 + \cos^2 x}{\cos x} dx = -\int \sec x\, dx + \int \cos x\, dx$

$= -\log|\sec x + \tan x| + \sin x$

$G(x) = \int g(x)dx = \int \sin x\, dx = -\cos x$

$\therefore PI = uF + vG = -\cos x[\log|\sec x + \tan x|] + \cancel{\sin x \cos x} - \cancel{\cos x \sin x}$.

$y = CF + PI = c_1\cos x + c_2\sin x - \cos x[\log|\sec x + \tan x|]$.

Q $\frac{d^2y}{dx^2} + n^2y = \sec nx$ $y = c_1\cos nx + c_2\sin nx + \frac{\cos nx \log(\cos nx)}{n^2} + \frac{x\sin nx}{n}$

Q $\frac{d^2y}{dx^2} + 4y = \sec^2 2x$ $y = c_1\cos 2x + c_2\sin 2x + \frac{\sin 2x}{4}[\log|\sec 2x + \tan 2x|] - \frac{1}{4}$.

Q $\frac{d^2y}{dx^2} + y = \cot x$ $y = c_1\cos x + c_2\sin x + \sin x[\log|\csc x - \cot x|]$.

Q $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = e^x$.

CF: $c_1e^x + c_2e^{2x} + c_3e^{3x}$

CF: $D^3 - 6D^2 + 11D - 6 = 0$ $D = 1,2,3$

$u = e^x$ $v = e^{2x}$ $w = e^{3x}$.

$W = \begin{vmatrix} u & v & w \\ u' & v' & w' \\ u'' & v'' & w'' \end{vmatrix} = 2e^{6x}$.

$f(x) = \frac{R}{W}\begin{vmatrix} v & w \\ v' & w' \end{vmatrix} = \frac{1}{2}$ $g(x) = -\frac{R}{W}\begin{vmatrix} u & w \\ u' & w' \end{vmatrix} = -e^{-x}$.

$h(x) = \frac{R}{W}\begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \frac{1}{2}e^{-2x}$

$F(x) = \int f(x)dx = x/2$

$G(x) = \int g(x)dx = e^{-x}$

$H(x) = \int h(x)dx. = -\frac{1}{4}e^{-2x}$

$PI = uF + vG + wH$

$= \frac{xe^x}{2} + e^x - \frac{1}{4}e^x = \frac{xe^x}{2} + \frac{3}{4}\cancel{e^x}$

$y = CF + PI$
$= c_1e^x + c_2e^{2x} + c_3e^{3x} + \frac{1}{2}xe^x$

Q $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$ $\quad u = e^x \quad v = e^{-x} \quad W = -2$

$f(x) = \frac{1}{e^x(1+e^x)} \quad g(x) = \frac{-e^x}{1+e^x}$

$F(x) = \int \frac{dx}{e^x(1+e^x)}$ Put $e^x = t$. $= \int \frac{dt}{t^2(1+t)} \quad \frac{1}{t^2(1+t)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{1+t}$

Put $t=0$, $t=1$ $\quad B=1, C=1 \quad A=-1$ $\quad \frac{At+B}{t^2} + \frac{C}{1+t}$

$G(x) = -\int \frac{e^x}{1+e^x}\,dx = -\log(1+e^x)$

$\therefore y = c_1 e^x + c_2 e^{-x} + e^x \log(1+e^{-x}) - e^{-x}\log(1+e^x) - 1.$

Q $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2e^x$ (Cauchy Eulers eq)

CF: let $z = \log x$ $\quad$ Aux eqn.

eq reduces to $(D(D-1) + D - 1)y = 0$ (To solve CF)

$(D^2-1)y = 0 \quad D = \pm 1. \quad$ CF: $c_1e^z + c_2e^{-z} = c_1x + c_2x^{-1}$.

Now $u = x \quad v = x^{-1}. \quad R = \frac{x^2e^x}{x^2}$. ($\because$ Dividing by x^2 we get in form of VOP)

$u' = 1 \quad v' = -\frac{1}{x^2}$

Now $y = CF + PI = c_1x + c_2x^{-1} + e^x(1 - x^{-1})$.

Q $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2\log x$. $\quad y = c_1x + c_2x^{-1} + \frac{1}{3}x^3\log x - \frac{4}{9}x^2$.

Q $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x\log x \quad y = c_1e^x + c_2xe^x - \frac{5}{36}x^3e^x + \frac{1}{6}x^3e^x\log x$.

Solving a diff eqn by reducing its order

Solving a diff eqn by reducing its order

Thm: Hypothesis: let f be a non trivial soln of 2nd order hom. LDE — ①

$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0$

Conclusion 1: The transformation $y = f(x).v$ reduces eqn ① into 1st order LDE $a_0(x)f(x)\frac{dw}{dx} + [2a_0(x)f'(x) + a_1(x)f(x)]w = 0$. ②

in the dep. var w, where $w = \frac{dv}{dx}$.

Conclusion 2: The particular soln $w = \frac{\exp\left[-\int \frac{a_1(x)}{a_0(x)}dx\right]}{[f(x)]^2}$

of eqn ② gives rise to funn v,

where $v(x) = \int \frac{\exp\left[-\int \frac{a_1(x)}{a_0(x)}dx\right]}{[f(x)]^2}\,dx$.

The funn g defined by $g(x) = f(x)v(x)$ is then a soln of 2nd order eqn ①. Conclusion 3: f & g are lin indep solns of ①. Hence gen soln $y = c_1f + c_2g$.

Q. Given that $y = x$ is a soln of $(x^2+1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$ —(1)
find a lin. indep soln by reducing its order.

Check: $y = x$ is a soln of (1)

Step 1 let $\boxed{y = xv}$ $f = x \to$ one soln

Then $\frac{dy}{dx} = x\frac{dv}{dx} + v$ & $\frac{d^2y}{dx^2} = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}$.

Then (1) becomes $(x^2+1)\left(x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}\right) - 2x\left(x\frac{dv}{dx} + v\right) + 2xv = 0$.

$\Rightarrow x(x^2+1)\frac{d^2v}{dx^2} + 2\frac{dv}{dx} = 0$.

Step 2: let $\boxed{w = \frac{dv}{dx}}$

Then we obtain the first-order hom. LDE (separable in w & x)

$$x(x^2+1)\frac{dw}{dx} + 2w = 0.$$

$$\Rightarrow \frac{dw}{w} = -\frac{2\,dx}{x(x^2+1)}$$

$$\Rightarrow \frac{dw}{w} = \left(\frac{-2}{x} + \frac{2x}{x^2+1}\right)dx \quad \text{(By partial fraction)}.$$

Integrating, we have (without adding constt.)

$$\log w = -2\log x + \log(x^2+1) + \cancel{\log} c$$

$$w = \frac{c(x^2+1)}{x^2}$$

Step 3:- By choosing $c = 1$, we recall that $\frac{dv}{dx} = w$ & integrate $\Rightarrow v = \int w\,dx$

$$v = \int\left(\frac{x^2+1}{x^2}\right)dx = x - \frac{1}{x}$$

Step 4:- let $g = f.v = x\left(x - \frac{1}{x}\right)$ is another soln.

$\therefore g(x) = x^2 - 1$.

where ~~Since~~ f & g are lin indep soln of (1). $\therefore$ general soln $y = c_1x + c_2(x^2-1)$

Q1 Given that $y = x$ is a soln of $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 4y = 0$
find a lin indep soln by reducing the order write the general soln. $[y = c_1x + c_2x^4]$

Q2 Given that $y = x+1$ is a soln of $(x+1)^2\frac{d^2y}{dx^2} - 3(x+1)\frac{dy}{dx} + 3y = 0$
find a lin. indep soln by reducing the order. Write the general soln.

Simultaneous Differential Equations

Step 1: Eliminate one of the dep var say y

Step 2: After eliminating, we get a LDE in x & t, solve it

Step 3: $x = \phi(t)$ is a sol^n of in step (2). Substitute x & $\frac{dx}{dt}$ in given eq^n to obtain $y = \psi(t)$

Step 4: $x = \phi(t)$ & $y = \psi(t)$ constitute the reqd. sol^n.

Q1 Solve
$$\frac{dx}{dt} - 7x + y = 0$$
$$\frac{dy}{dt} - 2x - 5y = 0.$$

Let $D \equiv \frac{d}{dt}$

Then above eq^n's become: $(D-7)x + y = 0$ — (1)

~~$(D-5)y - 2x = 0$.~~

$-2x + (D-5)y = 0$ — (2)

Multiply (1) by $(D-5)$: $(D^2-12D+35)x + (D-5)y = 0$

Multiply (2) by 1 (Keep as it is): $-2x + (D-5)y = 0$

$\quad + \qquad -$

Subtract: $(D^2-12D+37)x = 0$

$\Rightarrow D = 6 \pm i \quad (\because D \equiv \frac{d}{dt})$

CF: $e^{6t}(c_1\cos t + c_2\sin t)$

PI $= 0$ ($\because$ RHS is 0)

$\therefore x = e^{6t}(c_1\cos t + c_2\sin t)$ is sol^n.

Using (1) $y = 7x - \frac{dx}{dt}$

$$\frac{dx}{dt} = e^{6t}(-c_1\sin t + c_2\cos t) + 6e^{6t}(c_1\cos t + c_2\sin t)$$

$= \cancel{7c_1e^t - c_1e^t = 6c_1e^t}$.

$= 7e^{6t}c_1\cos t + 7e^{6t}c_2\sin t + e^{6t}c_1\sin t - e^{6t}c_2\cos t - 6e^{6t}c_1\cos t - 6e^{6t}c_2\sin t$.

$= c_1e^{6t}\cos t + c_2e^{6t}\sin t - e^{6t}c_2\cos t + c_1e^{6t}\sin t$

$$y = e^{6t}\left[(c_1-c_2)\cos t + (c_1+c_2)\sin t\right].$$

Q2 Solve: $\frac{dx}{dt} + 2x - 3y = t$ — (1) $\Rightarrow (D+2)x - 3y = t$. where $D \equiv \frac{d}{dt}$.

$\frac{dy}{dt} - 3x + 2y = e^{2t}$ — (2) $\Rightarrow -3x + (D+2)y = e^{2t}$.

Multiply (1) by $(D+2)$: $(D+2)^2x - 3(D+2)y = (D+2)t$

& (2) by -3: $-9x + 3(D+2)y = 3e^{2t}$.

$((D+2)^2 - 9)x = (D+2)t + 3e^{2t}$ ie. $(D^2+4D-5)x = 1+2t+3e^{2t}$

LDE of x wrt t

CF: $c_1e^t + c_2e^{-5t}$ $\quad (PI)_1 = -\frac{1}{5}$ $\quad (PI)_2 = -\frac{1}{5}(2t + \frac{4}{5}\cdot 2)$ $\quad (PI)_3 = \frac{3}{7}e^{2t}$

$x = c_1e^t + c_2e^{-5t} + \frac{3}{7}e^{2t} - \frac{1}{25}(10t+13)$ & $y = c_1e^t - c_2e^{-5t} + \frac{4}{7}e^{2t} - \frac{3}{5}t - \frac{12}{25}$.

Q3. Solve: $\frac{dx}{dt} + \frac{dy}{dt} + 2x + y = 0 \equiv (D+2)x + (D+1)y = 0$ — ①

$\frac{dy}{dt} + 5x + 3y = 0 \equiv 5x + (D+3)y = 0$ — ②

Multiply ① by $(D+3)$ & ② by $(D+1)$ & subtract $\therefore (D^2+1)x = 0$

$D = \pm i$

$x = c_1 \cos t + c_2 \sin t$.

$\frac{dx}{dt}$ Subtract ② from ① $\frac{dx}{dt} - 3x - 2y = 0$.

(else complicated)

$y = \frac{1}{2}(c_2 - 3c_1)\cos t$
$-\frac{1}{2}(c_1 + 3c_2)\sin t$.

Q4 Solve: $4\frac{dx}{dt} + 9\frac{dy}{dt} + 44x + 49y = t$

$3\frac{dx}{dt} + 7\frac{dy}{dt} + 34x + 38y = e^t$

$(4D+44)x + (9D+49)y = t$
$(3D+34)x + (7D+38)y = e^t$ $\Big\} \rightarrow (D^2+7D+6)x = 7 + 38t - 58e^t$

$x = -c_1 e^{-t} - 6c_2 e^{-6t} - \frac{29}{7}e^t + \frac{19}{3}$

$y = -c_1 e^{-t} + 4be^{-6t} + \frac{24}{7}e^t - \frac{17t}{3} + \frac{55}{9}$ (By eliminating $\frac{dy}{dt}$ (By sub.))

Q5 Solve: $\frac{d^2x}{dt^2} - 3x - 4y = 0$

$\frac{d^2y}{dt^2} + x + y = 0$

$(D^2-3)x - 4y = 0$ — ① $\times (D^2+1)$

$x + (D^2+1)y = 0$ — ② $\times$ 4

$(D^4 - 2D^2 + 1)x + \cancel{(\ldots)}y = 0$ (Add)

$D^4 - 2D^2 + 1 = 0 \Rightarrow (D^2-1)^2 = 0 \Rightarrow D^2 = 1, 1 \Rightarrow D = \pm 1, \pm 1$

$\therefore$ CF: $x = (c_1 + c_2 t)e^t + (c_3 + c_4 t)e^{-t}$

$\frac{dx}{dt} = \qquad \frac{d^2x}{dt^2} \qquad$ in ① $\Rightarrow 2y = (c_2 - c_1 - c_2 t)e^t - (c_3 + c_4 + c_4 t)e^{-t}$.

Q $\frac{dx}{dt} + 9\frac{dy}{dt} + 2x + 31y = e^t$ & $3\frac{dx}{dt} + 7\frac{dy}{dt} + x + 24y = 3$

Q $\frac{dx}{dt} + \frac{dy}{dt} + 2x + y = 0$ & $\frac{dy}{dt} + 5x + 3y = 0$

Q $\frac{dx}{dt} + 5x + y = e^t$ & $\frac{dy}{dt} - x + 3y = e^{2t}$.

Simultaneous eq^n of first order (Involving 3 vars.)

Of the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ $\qquad P, Q, R$ — fun^n of x, y, z.

Sol^n is of the Type $f(x,y,z) = c_1$ & $g(x,y,z) = c_2$.

Procedure : (1) Obtain 2 indep integrals by Equating two of above ratios or, by using Multipliers.

Q. Solve $\frac{x\,dx}{y^2 z} = \frac{dy}{xz} = \frac{dz}{y^2}$
$\qquad (1) \qquad (2) \qquad (3)$

(1) = (2) $\Rightarrow \frac{x\,dx}{y^2 \cancel{z}} = \frac{dy}{x\cancel{z}} \Rightarrow x^2 dx = y^2 dy \Rightarrow \frac{x^3}{3} - \frac{y^3}{3} = \text{const}$

$\Rightarrow x^3 - y^3 = c_1$

(2) = (3) does not work.

(1) = (3) $\Rightarrow \frac{x\,dx}{\cancel{y^2} z} = \frac{dz}{\cancel{y^2}} \Rightarrow x\,dx = z\,dz \Rightarrow x^2 - z^2 = c_2$

Sol^n $\quad x^3 - y^3 = c_1, \; x^2 - z^2 = c_2$

Q Solve $\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{nxy}$
$\qquad (1) \qquad (2) \qquad (3)$

(1) = (2) $\Rightarrow \frac{dx}{x^2} = \frac{dy}{y^2} \Rightarrow -\frac{1}{x} = -\frac{1}{y} + c_1 \Rightarrow \cancel{\frac{1}{x} - \frac{1}{y} = c_1}$ $\qquad \frac{1}{y} = \frac{1}{x} + c_1$ — (*)

(1) = (3) $\Rightarrow \frac{dx}{x^2} = \frac{dz}{nxy} \Rightarrow \frac{dx}{x} = \frac{1}{ny} dz \Rightarrow \frac{dx}{x} = \frac{1}{n}\left(\frac{1}{x} + c_1\right) dz$

$\Rightarrow \frac{\cancel{x}\,dx}{\cancel{x}(1 + c_1 x)} = \frac{1}{n} dz \Rightarrow \frac{\log(1 + c_1 x)}{c_1} = \frac{1}{n} z + c_2$

Put value of c_1
$\Rightarrow \frac{\log\left[1 + \left(\frac{1}{y} - \frac{1}{x}\right) x\right]}{\left(\frac{1}{y} - \frac{1}{x}\right)} = \frac{1}{n} z + \text{const}$

$\Rightarrow \log\left[\frac{x - \cancel{y} + \cancel{y}}{y}\right] = \frac{1}{n} z \left[\frac{x - y}{xy}\right] + \text{const}$

$\Rightarrow \log(x/y) + \frac{z}{n}\left(\frac{y - x}{xy}\right) = c_2$

Q Solve $\frac{dx}{y^2} = \frac{dy}{x^2} = \frac{dz}{x^2 y^2 z^2}$

(1) = (2) $\quad x^2 dx = y^2 dy \Rightarrow x^3 - y^3 = c_1$

(2) = (3) $\quad \frac{dy}{\cancel{x^2}} = \frac{dz}{\cancel{x^2} y^2 z^2} \Rightarrow y^2 dy = \frac{dz}{z^2}$

$\Rightarrow \frac{y^3}{3} = -\frac{1}{z} + c_2$

(8)

Q. Solve $\frac{dx}{xz(z^2+xy)} = \frac{dy}{-yz(z^2+xy)} = \frac{dz}{x^4}$

(1) = (2) $\Rightarrow \frac{dx}{x} = -\frac{dy}{y} \Rightarrow \log x + \log y = \log c_1 \Rightarrow xy = c_1$

(1) = (3) $\Rightarrow \frac{dx}{\cancel{x}z(z^2+xy)} = \frac{dz}{\cancel{x^4}x^3} \Rightarrow \frac{dx}{z(z^2+c_1)} = \frac{dz}{x^3} \Rightarrow x^3\,dx = (z^3 + c_1 z)\,dz$

$\boxed{\text{(2) = (3) can also be used.}}$

$\Rightarrow x^4 = z^4 + 2c_1 z^2 + c_2$

$\Rightarrow x^4 = z^4 + 2xyz^2 + c_2$.

Using Multipliers

Consider $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = k$ (say).

Find multipliers $\lambda_1, \lambda_2, \lambda_3$ for P, Q, R resp so that

$\lambda_1 P + \lambda_2 Q + \lambda_3 R = 0$. $\boxed{\lambda_1, \lambda_2, \lambda_3 \to \text{may involve } x, y, z \text{ or consts}}$

Now $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{\lambda_1 dx + \lambda_2 dy + \lambda_3 dz}{(\lambda_1 P + \lambda_2 Q + \lambda_3 R) = 0} = k$.

$\Rightarrow \lambda_1 dx + \lambda_2 dy + \lambda_3 dz = k.0 = 0$

$\Rightarrow \int \lambda_1 dx + \int \lambda_2 dy + \int \lambda_3 dz = \text{constt}$.

Q Solve $\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)}$.

Taking $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers:

$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\frac{1}{x}.x(y^2-z^2) + \frac{1}{y}.y(z^2-x^2) + \frac{1}{z}.z(x^2-y^2)} = \frac{(\ )}{0}$

$\therefore \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0. \Rightarrow \log x + \log y + \log z = \log c_1 \Rightarrow xyz = c_1$

Taking x, y, z: $x[x(y^2-z^2)] + y[y(z^2-x^2)] + z[z(x^2-y^2)] = 0$.

where $c_1, c_2 \to$ arb. constts.

$\therefore x\,dx + y\,dy + z\,dz = 0 \Rightarrow x^2 + y^2 + z^2 = c_2$

Q Solve $\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$.

$\begin{vmatrix} l & m & n \\ l & m & n \\ x & y & z \end{vmatrix} = 0$

Taking l, m, n

RHS $\Rightarrow lx + my + nz = c_1$

$\Rightarrow l(mz - ny) + m(nx - lz) + n(ly - mx)$.

$\Rightarrow x^2 + y^2 + z^2 = c_2$

RHS $= \begin{vmatrix} x & y & z \\ l & m & n \\ x & y & z \end{vmatrix} = 0$

Taking x, y, z

Q Solve $\frac{a\,dx}{(b-c)yz} = \frac{b\,dy}{(c-a)zx} = \frac{c\,dz}{(a-b)xy}$

$x, y, z \Rightarrow ax^2 + by^2 + cz^2 = c_1$

$ax, by, cz \Rightarrow a^2x^2 + b^2y^2 + c^2z^2 = c_2$

Single diff. eqⁿs that are integrable (Total Diff Eqⁿ).

The eqⁿ $Pdx + Qdy + Rdz = 0$ — (1) has an integral $u = a$ — (2) when ∃ funⁿ u whose total differential du is equal to (1) or to a multiple of (1).

If (1) have an integral (2), then since

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

P, Q, R must be proportional to $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$

i.e. $\frac{\partial u}{\partial x} = \mu P$, $\frac{\partial u}{\partial y} = \mu Q$, $\frac{\partial u}{\partial z} = \mu R$

These 3 condⁿs can be reduced to one involving the coeffs P, Q, R & their derivatives

Diff. 1st of these 3 eqⁿ wrt y & z, 2nd wrt z & x, 3rd wrt x & y

$$P\frac{\partial \mu}{\partial y} + \mu\frac{\partial P}{\partial y} = \frac{\partial^2 u}{\partial x \partial y} = Q\frac{\partial \mu}{\partial x} + \mu\frac{\partial Q}{\partial x}$$

$$Q\frac{\partial \mu}{\partial z} + \mu\frac{\partial Q}{\partial z} = \frac{\partial^2 u}{\partial y \partial z} = R\frac{\partial \mu}{\partial y} + \mu\frac{\partial R}{\partial y}$$

$$R\frac{\partial \mu}{\partial x} + \mu\frac{\partial R}{\partial x} = \frac{\partial^2 u}{\partial z \partial x} = P\frac{\partial \mu}{\partial z} + \mu\frac{\partial P}{\partial z}.$$

On rearranging,

$$\left(\mu\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = Q\frac{\partial \mu}{\partial x} - P\frac{\partial \mu}{\partial y} \right) \times R$$

$$\left(\mu\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) = R\frac{\partial \mu}{\partial y} - Q\frac{\partial \mu}{\partial z} \right) \times P$$

$$\left(\mu\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) = P\frac{\partial \mu}{\partial z} - R\frac{\partial \mu}{\partial x} \right) \times Q$$

& add.

$$\boxed{P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0} \quad (3)$$

The relⁿ (3) must exist b/w the coeffs of (1) when it is integrable.

Converse, also true

∴ (3) is nec & suff condⁿ that (1) is Integrable

(1) To solve by inspection.

Q $(y+z)dx + dy + dz = 0$ check cond^n.

$\Rightarrow dx + \dfrac{dy+dz}{(y+z)} = 0 \Rightarrow x + \log(y+z) = c$

Q $zy\,dx = zx\,dy + y^2dz \Rightarrow \dfrac{y\,dx - x\,dy}{y^2} = \dfrac{dz}{z} \Rightarrow d\left(\dfrac{x}{y}\right) = \dfrac{d}{z}$

$\Rightarrow z(y\,dx - x\,dy) = y^2dz$

$\Rightarrow \dfrac{x}{y} = \log z + c$.

~~**Q** $(2x^2 + 2xy + 2xz^2 + 1)dx + dy + 2z\,dz = 0$.~~

(2) Homogeneous.

Q $(yz+z^2)dx - xz\,dy + xy\,dz = 0$ — (1) check cond^n

$P = yz + z^2 \quad Q = -xz \quad R = xy$.

(1) is hom ∴ let $x = uz$, $y = vz$

$dx = u\,dz + z\,du \qquad dy = v\,dz + z\,dv$

Put in (1), $(v+1)z^2(u\,dz + z\,du) - uz^2(v\,dz + z\,dv) + uvz^2dz = 0$

$\Rightarrow (v+1)z^3du - uz^3dv + \left[(v+1)uz^2 - uvz^2 + uvz^2\right]dz = 0$

$\Rightarrow$ Dividing by $(v+1)uz^3$

$\dfrac{du}{u} - \dfrac{dv}{v+1} + \dfrac{dz}{z} = 0 \Rightarrow \log u - \log(v+1) + \log z = \log c$.

$\Rightarrow \dfrac{uz}{v+1} = c \Rightarrow \dfrac{\frac{x}{z}\cdot z}{\frac{y}{z}+1} = c$.

$\Rightarrow \dfrac{xz}{y+z} = c \Rightarrow \boxed{xz = c(y+z)}$

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Q9 $(y^2+yz)dx + (xz+z^2)dy + (y^2-xy)dz = 0$.

condⁿ of Int

$x = uz \quad y = vz$.

(1) $\Rightarrow (v^2z^2 + vz^2)(u\,dz + z\,du) + (uz^2+z^2)(v\,dz + z\,dv) + (v^2z^2 - uvz^2)dz$

$\Rightarrow (v^2 + v)z^3du + (u+1)z^3dv + v^2uz^2dz + uvz^2dz + \cancel{uvz^2dz}$

$+ vz^2dz + v^2z^2dz - \cancel{uvz^2dz} = 0$

$= (u+1)v^2z^2dz + (u+1)vz^2dz$

$= (u+1)(v^2 + v)z^2dz = 0$.

Divide by $(u+1)(v^2+v)z^3$

$\Rightarrow \dfrac{du}{u+1} + \dfrac{dv}{v^2+v} + \dfrac{dz}{z} = 0$

→ Partial fraction -

$\dfrac{1}{v^2+v} = \dfrac{1}{v(v+1)} = \dfrac{A}{v} + \dfrac{B}{v+1}$

$1 = A(v+1) + Bv$

~~$v=0 \Rightarrow A=1$~~

~~$v=1 \Rightarrow 2A+B=1$~~

$1 = A$ $\quad A+B=0$, $B=-1$

$\Rightarrow \dfrac{du}{u+1} + \left(\dfrac{1}{v} - \dfrac{1}{v+1}\right)dv + \dfrac{dz}{z} = 0$

$\Rightarrow \log(u+1) + \log v - \log(v+1) + \log z = \log c$.

$\Rightarrow \dfrac{(u+1)(vz)}{(v+1)} = c \Rightarrow \dfrac{\left(\frac{x}{z}+1\right)y}{\left(\frac{y}{z}+1\right)} = c$

$\Rightarrow \dfrac{(x+z)y}{y+z} = c \Rightarrow (x+z)y = c(y+z)$.

Q $z(z-y)dx + z(z+x)dy + x(x+y)dz = 0$. Solⁿ: $z(x+y) = c(x+z)$.

(Try $y = ux$, $z = vx$).

Type: Non-homogeneous T.D.E. (Applicable to hom also) —— (1)

$P\,dx + Q\,dy + R\,dz = 0$

Step (1). condⁿ of Int.

Take any one var as constt. say z = constt

then $dz = 0$.

Step 2

∴ (1) reduces to $P\,dx + Q\,dy = 0$ —— (2)

Integrate (2) (keeping z constt) to get $f(x,y,z) =$ constt.

Step 3 let $V = f(x,y,z)$

$dV = \dfrac{\partial V}{\partial x}dx + \dfrac{\partial V}{\partial y}dy + \dfrac{\partial V}{\partial z}dz$

Use (1) to reduce dV in terms of dz only

Step 4 Solve V in terms of z.

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Q. $3x^2dx + 3y^2dy - (x^3+y^3+e^{2z})dz = 0$ —— (1)

Step 1: cond^n

Step 2: let $z = $ constt $\therefore dz = 0$

(1) reduces to $3x^2dx + 3y^2dy = 0 \Rightarrow x^3+y^3 = $ constt.

Step 3: let $V = x^3+y^3$ —— (2)

$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$

$= 3x^2dx + 3y^2dy$

$= (x^3+y^3+e^{2z})dz$ (Using (1))

$= (V+e^{2z})dz$ (Using (2))

Step 4: $\frac{dV}{dz} = V + e^{2z}$.

$\Rightarrow \frac{dV}{dz} - V = e^{2z}$ (linear in V & z)

$\Rightarrow$ IF $= e^{\int -dz} = e^{-z}$

$\Rightarrow Ve^{-z} = \int e^{2z}.e^{-z}dz = \int e^{z}dz \Rightarrow V = e^{2z} + ce^{z}$

$\Rightarrow x^3+y^3 = e^{2z} + ce^{z}$,

Q. $(2x^2 + 2xy + 2xz^2 + 1)dx + dy + 2z\,dz = 0$. —— (1)

Step 1 cond^n

Step 2: let $x = $ constt [$\because$ dx coeff worst]

$\therefore dx = 0$ ie $dy + 2z\,dz = 0 \Rightarrow y + z^2 = $ constt

Step 3: $V = y + z^2$ / $dV = dy + 2z\,dz = -\{2x^2+2xy+2xz^2+1\}dx$ (By (1))

$= -(2x^2 + 2xV + 1)dx$

$\Rightarrow \frac{dV}{dx} = -2x^2 - 2xV - 1 \Rightarrow \frac{dV}{dx} + 2x.V = -2x^2 - 1$ (linear)

IF $= e^{\int 2x\,dx} = e^{x^2}$.

$\therefore Ve^{x^2} = \int(-2x^2-1)e^{x^2}dx. = -2\int x^2e^{x^2}dx - \int e^{x^2}dx$. —— (*)

Consider $\int x^2e^{x^2}dx = \int x(xe^{x^2})dx = x\int xe^{x^2}dx - \int\{\int xe^{x^2}dx\}dx$ —— (**)

Consider $\int xe^{x^2}dx$ let $x^2 = t$

$= \int e^t.\frac{dt}{2} = \frac{1}{2}e^{x^2}$.

(**) $\Rightarrow \int x^2e^{x^2}dx = x.\frac{1}{2}e^{x^2} - \int \frac{1}{2}e^{x^2}dx$

(*) $\Rightarrow Ve^{x^2} = -2[x.\frac{1}{2}e^{x^2} - \int \frac{1}{2}e^{x^2}dx] - \int e^{x^2}dx$

$$V \cdot e^{x^2} = -x e^{x^2} + c$$

$$\Rightarrow V = -x + c e^{-x^2}$$

$$\Rightarrow y + z^2 = -x + c e^{-x^2}$$

$$\Rightarrow x + y + z^2 = c e^{-x^2}.$$

Q $(x^2+y^2+z^2)dx - 2xy\,dy - 2xz\,dz.$ Sol^n $\frac{y^2+z^2}{x} = x + c.$