## NORMAL DISTRIBUTION

### References:

[1] MILLER & FREUND'S, PROBABILITY AND STATISTICS FOR ENGINEERS, NINTH EDITION ,Richard A. Johnson, Pearson Ed., 2018

[2] http://www.universityofcalicut.info/SDE/Probability_distribn_4june2015.pdf

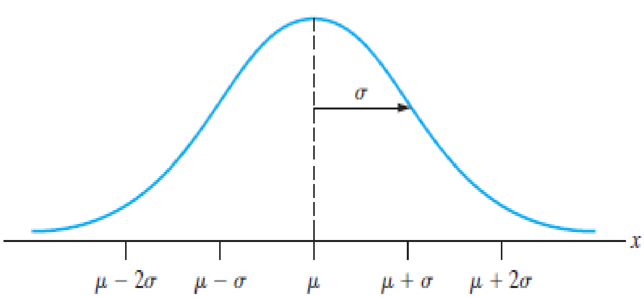
### Definition.

A continuous random variable X with pdf $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{(x-\mu)^2}{2\sigma^2}}, \ \infty < x < \infty$ is said to follow normal distribution with parameters μ and σ, denoted by X ~ N(μ,σ).

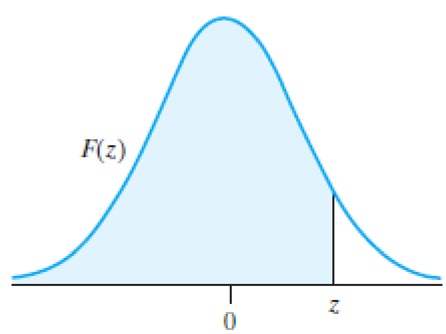
## Standard Normal Distribution

A normal distribution with mean μ = 0 and standard deviation σ = 1 is called a standard normal distribution. If Z is a standard normal variable then its pdf is,

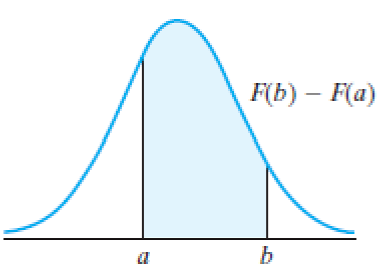
$$f(z) = \frac{1}{\sqrt{(2\pi)}} e^{\frac{-z^2}{2}}, -\infty < z < \infty$$




Graph of normal probability density

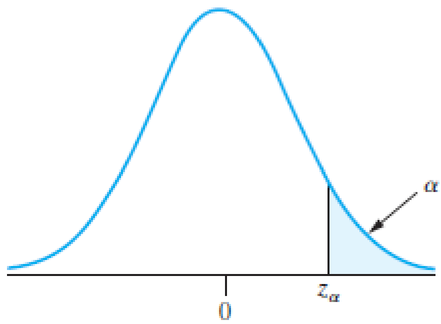



The standard normal probabilities $F(z) = P(Z \leq z)$



The standard normal probability $F(b) - F(a) = P(a < Z \le b)$

Let $z_\alpha$ be such that the probability is $\alpha$ that it will be exceeded by a random variable having the standard normal distribution. That is, $\alpha = P(Z > z_\alpha)$ as illustrated




### Two important values for $z_\alpha$

Find (a) $z_{0.01}$; (b) $z_{0.05}$.

**(a)** Since $F(z_{0.01}) = 0.99$, we look for the entry in Table 3 which is closest to 0.99 and get 0.9901 corresponding to $z = 2.33$. Thus $z_{0.01} = 2.33$.

**(b)** Since $F(z_{0.05}) = 0.95$, we look for the entry in Table 3 which is closest to 0.95 and get 0.9495 and 0.9505 corresponding to $z = 1.64$ and $z = 1.65$. Thus, by interpolation, $z_{0.05} = 1.645$. ■

When $X$ has the normal distribution with mean $\mu$ and standard deviation $\sigma$.

$$P(a < X \le b) = F\left(\frac{b-\mu}{\sigma}\right) - F\left(\frac{a-\mu}{\sigma}\right)$$

**EXAMPLE.**

Find the probabilities that a random variable having the standard normal distribution will take on a value

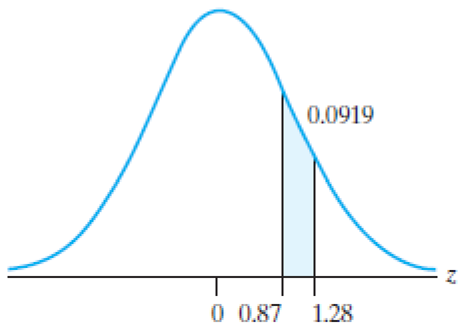
**(a)** between 0.87 and 1.28;

**(b)** between −0.34 and 0.62;

**(c)** greater than 0.85;

**(d)** greater than −0.65.

**Solution. Part (a)**

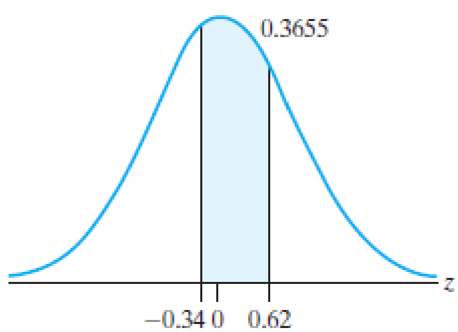



$P(0.87 < Z < 1.28)$

Looking up the necessary values in Table 3, for part (a) we get

$$F(1.28) - F(0.87) = 0.8997 - 0.8078 = 0.0919$$

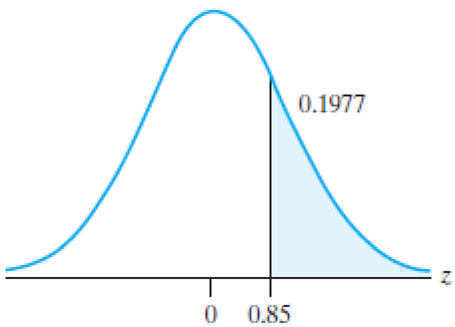
**Part (b)**




$P(-0.34 < Z < 0.62)$

$$F(0.62) - F(-0.34) = 0.7324 - 0.3669 = 0.3655$$

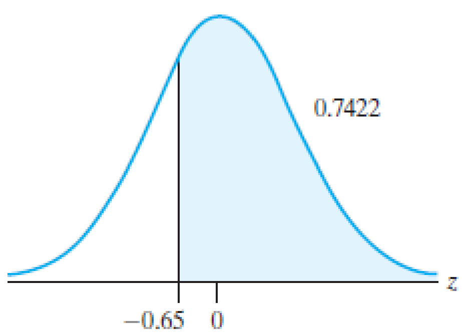
**Part (c )**




$P(Z > 0.85)$

$$1 - F(0.85) = 1 - 0.8023$$
$$= 0.1977$$

**Part (d)**




$P(Z > -0.65)$

$$1 - F(-0.65) = 1 - 0.2578 = 0.7422$$

or, alternatively,

$$1 - F(-0.65) = 1 - [1 - F(0.65)]$$
$$= F(0.65)$$
$$= 0.7422$$

**Mean and Variance:**
Let X ~ N(μ,σ) ,then,

$$E(X)= \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \int_{-\infty}^{\infty}[(x-\mu)] +\mu] \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \int_{-\infty}^{\infty}[(x-\mu)] \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx + \int_{-\infty}^{\infty}[\mu] \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty}[(x-\mu)]\ e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx + \mu \int_{-\infty}^{\infty} f(x)dx$$

Put $\frac{x-\mu}{\sigma} = \mu \Rightarrow dx = \sigma du$

$$\Rightarrow E(X) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma u e^{\frac{\mu^2}{2}} \sigma du + \mu \times 1$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \times 0 + \mu = \mu$$

(Since $ue\frac{\mu^2}{2}$is an odd function of $u$, $\int_{-\infty}^{\infty} ue\frac{\mu^2}{2}\sigma du = 0$)
There fore mean = μ

$$V(X) = E(X - E(X))^2$$

$$= E(X - \mu)^2$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 \ \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Put $\frac{x-\mu}{\sigma} = z$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z)^2 e^{\frac{-z^2}{2}} \sigma dz$$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{\frac{-z^2}{2}} \sigma dz$$

Put $\frac{z^2}{2} = u$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2ue^{2-u} \frac{du}{\sqrt{2u}}$$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^{\frac{1}{2}} e^{-u} du$$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^{\frac{3}{2}-1} e^{-u} du$$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \frac{\Gamma\frac{3}{2}}{1^{\frac{3}{2}}}$$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \frac{1}{2} \Gamma\frac{1}{2}$$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \frac{1}{2} \sqrt{\pi} = \sigma^2$$

i.e., $V(X) = \sigma^2$

There fore Standard Ddeviation = $\sqrt{V(X)} = \sigma$

**Odd order moments about mean:**

$$\mu_{2r+1} = \int_{-\infty}^{\infty} (x-\mu)^{2r+1} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z)^{2r+1} e^{-\frac{-z^2}{2}} \sigma dz$$

By putting $\frac{x-\mu}{\sigma} = z$

$$= \frac{\sigma^{2r+1}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2r+1} e^{\frac{-z^2}{2}} dz$$

$$= \frac{\sigma^{2r+1}}{\sigma\sqrt{2\pi}} \times 0 = 0$$

Since the integrand is an odd function i.e., $\mu_{2r+1} = 0$, r = 0, 1,2, …

## Even order central moments:

$\mu_{2r} = 1.3.5...(2r - 1)\sigma^{2r}$

$\mu_{2r} = E(x - \mu)^{2r}$

$= \int_{-\infty}^{\infty} (x-\mu)^{2r} f(x)dx$

$= \int_{-\infty}^{\infty} (x-\mu)^{2r} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$

Put $\frac{x-\mu}{\sigma} = z$

$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z)^{2r} e^{\frac{-z^2}{2}} \sigma dz$

Put $\frac{x-\mu}{\sigma} = z$

$= \frac{\sigma^{2r}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (z)^{2r} e^{\frac{-z^2}{2}} dz,$

$= \frac{\sigma^{2r}}{\sqrt{2\pi}} \int_{0}^{\infty} (z)^{2r} e^{\frac{-z^2}{2}} dz,$

$Put \quad \frac{-z^2}{2} = u$

$$= \frac{2\sigma^{2r}}{\sqrt{2\pi}} \int_0^{\infty} (2u)^{2r} e^{-u} \frac{du}{\sqrt{2u}}$$

$$= \frac{2^r\sigma^{2r}}{\sqrt{2\pi}} \int_0^{\infty} (u)^{r-\frac{1}{2}} e^{-u} du$$

$$= \frac{2^r\sigma^{2r}}{\sqrt{2\pi}} \int_0^{\infty} (u)^{r+\frac{1}{2}} e^{-u} du$$

$$= \frac{2^r\sigma^{2r}}{\sqrt{\pi}} \frac{\Gamma(r+\frac{1}{2})}{1^{r+\frac{1}{2}}}$$

$$= \frac{2^r\sigma^{2r}}{\sqrt{\pi}} (r - \tfrac{1}{2})(r - \tfrac{3}{2}), \ldots, \frac{3}{2}, \frac{1}{2} \Gamma \frac{1}{2}$$

$$= \frac{2^r\sigma^{2r}}{\sqrt{\pi}} \frac{(2r-1)(2r-2)\ldots 3.1.\sqrt{\pi}}{2^r}$$

$$\mu_{2r} = 1.3.5\ldots(2r - 1)\sigma^{2r}$$

## Recurrence relation for even order central moments

We have $\mu_{2r} = 1.3.5\ldots(2r - 1)\sigma^{2r}$

$$\mu_{2r+2} = 1.3.5\ldots(2r - 1)(2r + 1)\sigma^{2r+2}$$

There fore,

$$\frac{\mu_{2r+2}}{\mu_{2r}} = (2r + 1)\sigma^2$$

i.e.,

$$\mu_{2r+2} = (2r + 1)\sigma^2 \mu_{2r}$$

This is the recurrence relation for even order central moments of Normal Distribution. Using this relationship we can find out the 2nd and 4th moments.
Put r = 0 then $\mu_2 = \sigma_2$

$r = 1 \Rightarrow \mu_4 = 3\sigma^4$

Since $\mu_3 = 0 \Rightarrow \beta_1 = 0, \gamma_1 = 0$

Also, $\beta_1 = \frac{\mu_4}{\mu_2^2} = 3$ and $\gamma_2 = 0$

**Moment generating function:**

$$Mx(t) = E(e^{tx})$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-2)^2}{2\sigma^2}} dx$$

Put $\frac{x-\mu}{\sigma} = z$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)} e^{\frac{-z^2}{2}} \sigma dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}(z^2-2t\sigma z+t^2\sigma^2)+\frac{1}{2}t^2\sigma^2} dz$$

$$= \frac{e^{\mu t+\frac{1}{2}t^2\sigma^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}(z-t\sigma)^2} dz$$

Put z-t$\sigma$ =$u$

$$= \frac{e^{\mu t+\frac{1}{2}t^2\sigma^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-u^2}{2}} du$$

$$= \frac{e^{\mu t+\frac{1}{2}t^2\sigma^2}}{\sqrt{2\pi}} 2\int_{-\infty}^{\infty} e^{\frac{-u^2}{2}} du$$

Put $\frac{u^2}{2} = v$

$$= \frac{e^{\mu t+\frac{1}{2}t^2\sigma^2}}{\sqrt{2\pi}} 2 \int_{0}^{\infty} e^{-v} \frac{dv}{\sqrt{2v}}$$

$$= \frac{e^{\mu t+\frac{1}{2}t^2\sigma^2}}{\sqrt{\pi}} \int_{0}^{\infty} v^{\frac{1}{2}-1} e^{-v} dv$$

$$= \frac{e^{\mu t+\frac{1}{2}t^2\sigma^2}}{\sqrt{\pi}} \frac{\Gamma\frac{1}{2}}{1\frac{1}{2}}$$

$$= \frac{e^{\mu t + \frac{1}{2} t^2 \sigma^2}}{\sqrt{\pi}} \sqrt{\pi}$$

Thus,

$$\text{Mx}(t) = e^{\mu t + \frac{1}{2} t^2 \sigma^2}$$

**TO OBTAIN MEAN, VARIANCE.. THROUGH MGF**

To obtain the moments of the normal, we differentiate once to obtain

$$M'(t) = e^{t\mu + \frac{1}{2}t^2\sigma^2}(\mu + t\sigma^2)$$

and a second time to get

$$M''(t) = e^{t\mu + \frac{1}{2}t^2\sigma^2}[(\mu + t\sigma^2)^2 + \sigma^2].$$

Setting $t = 0$,

$$E[X] = M'(0) = \mu \qquad \text{and} \qquad E(X^2) = M''(0) = \sigma^2 + \mu^2$$

so $Var(X) = \sigma^2$ as the notation suggests. ■

**Additive property:**

Let $X_1 \sim N(\mu_1, \sigma_1)$, $X_2 \sim N(\mu_2, \sigma_2)$ and if $X_1$ and $X_2$ are independent, then

$$X_1 + X_2 \sim N(\mu_1 + \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2}$$

**Proof:**

We have the mgf's of $X_1$ and $X_2$ are respectively,

$$M_{X1}(t) = e^{\mu 1t + \frac{1}{2}t^2 \sigma 1_1^2}$$

and

$$M_{X2}(t) = e^{\mu 2t + \frac{1}{2}t^2 \sigma 1_2^2}$$

Since $X_1$ and $X_2$ are independent

$$M_{X1+X2}(t) = M_{X_1}(t) + M_{X_2}(t)$$

$$= e^{\mu 1t + \frac{1}{2}t^2 \sigma 1_1^2} \times e^{\mu 2t + \frac{1}{2}t^2 \sigma 1_2^2}$$

$$= e^{(\mu 1 + \mu)t + \frac{1}{2}t^2(\sigma_1^2 + \sigma_2^2)}$$

i.e.,

$$X_1 + X_2 \sim N(\mu_1 + \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2}$$

**Remarks 1**

If $X_1, X_2, \ldots, X_n$ are n independent normal variates with mean = $\mu_i$ and variance = $\sigma_i^2$, ,i=1,2,...,$n$ respectively. Then the variate Y = $\sum_{i=n}^{n} Xi$ is normally distributed with mean = $\sum_{i=n}^{n} \mu i$ and variance = $\sum_{i=n}^{n} \sigma_i^2$.

**Remarks 2**

If $X_1, X_2, \ldots, X_n$ are n independent normal variates with mean = $\mu_i$ and variance = $\sigma_i^2$, ,i=1,2,...,$n$ respectively. Then the variate Y = $\sum_{i=n}^{n} ai\, Xi$ is normally distributed with mean = $\sum_{i=n}^{n} a_i \mu_i$ and variance = $\sum_{i=n}^{n} a_i^2 \sigma_1^2$,where $a_i$'s are constants.

**Theorem 5.3** Let $X$ have moment generating function $M(t)$ and let $a$ and $b$ be constants. Then

$$M_{a+bX}(t) = E\left(e^{(a+bX)t}\right) = e^{at} \cdot M(bt)$$

For instance, the moment generating function of $X - \mu$, corresponding to $b = 1$ and $a = -\mu$, is

$$M_{X-\mu}(t) = e^{-\mu t} \cdot M_X(t)$$

For the **Standard Normal Distribution** ,

## Moment generating function:

$$M_Z(t) = M_{\frac{x-\mu}{\sigma}}(t) = e^{\frac{-\mu t}{\sigma}} M_{X(\frac{t}{\sigma})}$$

$$= e^{\frac{-\mu t}{\sigma}} \times e^{\frac{\mu t}{\sigma} + \frac{t^2\sigma^2}{\sigma^2}} = e^{\frac{1}{2}t^2}$$

## Normal distribution as a limiting form of binomial distribution

Binomial distribution tends to normal distribution under the following conditions

(i). n is very large ($n \to \infty$)

(ii). neither $p$ nor $q$ is very small.

**Proof:**

let X $\rightarrow$ B($n$,$p$)

Then

$$f(x) = {}^{n}C_{x} p^{x} q^{n-x}, x = 0, 1, 2, \ldots n$$

Also,

$$E(X) = np, V(X) = npq, M_X(t) = (q+pe^{t})^{n}$$

Define

$$Z = \frac{X-E(X)}{\sqrt{V(X)}} = \frac{X-np}{\sqrt{npq}} = \frac{x-\mu}{\sigma}$$

Now

$$M_Z(t) = M_{\frac{X-\mu}{\sigma}}(t)$$

$$= e^{\frac{-\mu t}{\sigma}} M_x(t/\sigma)$$

$$= e^{\frac{-\mu t}{\sigma}} (q+pe^{t/\sigma})^{n}$$

Then,

$$\log M_Z(t) = \frac{-\mu t}{\sigma} + n\log(q+pe^{t/\sigma})$$

$$= \frac{-\mu t}{\sigma} + n\log(q+pe^{t/\sigma})$$

$$= \frac{-\mu t}{\sigma} + n\log\left[ q + p\left(1 + \frac{t/\sigma}{1!} + \frac{(t/\sigma)^2}{2!} + \ldots\ldots\right)\right]$$

$$= \frac{-\mu t}{\sigma} + n\log\left[ q + p + p\left(\frac{t/\sigma}{1!} + \frac{(t/\sigma)^2}{2!} + \ldots\ldots\right)\right]$$

$$= \frac{-\mu t}{\sigma} + n\log\left[ 1 + p\left(\frac{t/\sigma}{1!} + \frac{(t/\sigma)^2}{2!} + \ldots\ldots\right)\right]$$

$$= \frac{-\mu t}{\sigma} + n\left[ p\left(\frac{t/\sigma}{1!} + \frac{(t/\sigma)^2}{2!} + \ldots\ldots\right) - \frac{p^2}{2}\left(\frac{t/\sigma}{1!} + \frac{(t/\sigma)^2}{2!} + \ldots\ldots\right)^2 + \ldots\ldots \right]$$

$$= \frac{-\mu t}{\sigma} + n\left[ \frac{pt}{\sigma} + \frac{pt^2}{2\sigma^2} - \frac{p^2t^2}{2\sigma^2} + 0\left(\frac{1}{n^{\frac{3}{2}}}\right)\right.$$

$$= \frac{-\mu t}{\sigma} + \frac{npt}{\sigma} + \frac{npt^2}{s\sigma^2}(1-p) + 0\left(\frac{1}{n^{\frac{1}{2}}}\right)$$

$$= \frac{t^2}{2} + 0\left(\frac{1}{n^{\frac{1}{2}}}\right) \to \frac{t^2}{2} \text{ as } n\to\infty$$

There fore

$$M_z(t) = e^{\frac{1}{2}t^2}$$

This is the mgf of a standard normal variate. So Z→ N(0, 1) i.e.,

$$= \frac{X-np}{\sqrt{npq}} \to N(0, 1) \text{ as } n\to\infty$$

$$X \to N(np, \sqrt{npq})$$

when n is very large.

**Example.**

IF X ~ N(12,4). Find

(i). $P(X \geq 20)$

(ii). $P(0 \leq X \leq 12)$

(iii). Find $a$ such that $P(X > a) = 0.24$.

**Solution.**

We have $Z = \frac{X-\mu}{\sigma} = \frac{X-12}{4} \sim N(0,1)$

(i)

$$P(X\geq 20) = P\left(\frac{X-12}{4} \geq \frac{20-12}{4}\right) = P(Z \geq 2)$$

$$=0.5-P(0<2)=0.5- 0.4772 =00228$$

(ii)

$$P(0 \leq X \leq 12) = P\left(\frac{0-12}{4} \leq \frac{X-12}{4} \leq \frac{12-12}{4}\right)$$

$$=P(-3\leq Z\leq 0)= P(0 \leq Z \leq 3)= 0.4987.$$

(iii)

$$\text{Given } P(X>a) = 0.24 \Rightarrow P\left(\frac{X-12}{4} > \frac{a-12}{4}\right)=0.24$$

$$\Rightarrow P\left( Z > \frac{a-12}{4}\right) = 0.24$$

$$\text{Hence } P\left( 0 < Z < \frac{a-12}{4}\right)=0.5 -0.24 =0.26$$

From a Standard Normal table the value of $\frac{a-12}{4}=0.71 \Rightarrow a =14.84$

**Example.**

Find k , if $P(X \le k) = 2P(X > k)$ where $X \sim N(\mu, \sigma)$

**Solution.**

Given that

$$P(X \le k) = 2P(X > k)$$

$$\Rightarrow \frac{P(X \le k)}{P(X > k)} = 2$$

$$\Rightarrow \frac{P(X \le k)}{P(X > k)} + \frac{P(X > k)}{P(X > k)} = 2+1$$

$$\Rightarrow \frac{P(X\ k) + P(X > k)}{P(X > k)} = 3$$

$$\Rightarrow \frac{1}{P(X > k)} = 3$$

$$\Rightarrow P(X > k) = \frac{1}{3} = 0.333$$

$$P\left[\frac{X-\mu}{\sigma} > \frac{k-\mu}{\sigma}\right] = 0.333$$

$$\text{i.e., } P\left(Z > \frac{K-\mu}{\sigma}\right) = 0.333$$

From table $\frac{k-\mu}{\sigma} = 0.44$

Then $k = \mu + 0.44\,\sigma$

**EXAMPLE.**

If X is a normal random variable with mean 6 and variance 49 and if $P(3X + 8 \leq \lambda) = P(4X - 7 \geq \mu)$ and $P(5X - 2 \leq \mu) = P(2X + 1 \geq \mu)$, find $\lambda$ and $\mu$.

**Solution.**

Given $X \sim N(6, 7)$

$$P(3X + 8 \geq \lambda) = P(4X - 7 \geq \mu)$$

$$\Rightarrow P\left(X \geq \frac{\lambda-8}{3}\right) = P\left(X \geq \frac{\mu+7}{4}\right) \text{ - - - - - - - - - (1)}$$

$$P(5X - 2 \leq \mu) = P(2X + 1 \geq \mu)$$

$$\Rightarrow P\left(X \leq \frac{\mu+2}{5}\right) = P\left(X \geq \frac{\lambda-1}{2}\right) \text{ - - - - - - - - - (2)}$$

Since

$$X \sim N(6,7),\ Z = \frac{X-6}{7} \sim N(0,1)$$

From(1), $$P\left(\frac{X-6}{7} \leq \frac{\frac{\lambda-8}{3}-6}{7}\right) = P\left(\frac{X-6}{7} \geq \frac{\frac{\mu+7}{4}-6}{7}\right)$$

$$\Rightarrow P\left(Z \leq \frac{\lambda-26}{21}\right) = P\left(Z \geq \frac{\mu-17}{28}\right)$$

From the standard normal curve, *if* $P(Z \leq a) = P(Z \geq b)$, then $a = -b$

That is

$$\frac{\lambda-26}{21} = -\left(\frac{\mu-17}{28}\right)$$

$$\Rightarrow 4\lambda + 3\mu - 155 = 0 \text{ - - - - - - - - - - (3)}$$

From (2) $$P\left(\frac{X-6}{7} \leq \frac{\frac{\mu+2}{5}-6}{7}\right) = P\left(\frac{X-6}{7} \geq \frac{\frac{\lambda-1}{2}-6}{7}\right)$$

$$\Rightarrow P\left(Z \leq \frac{\mu-28}{35}\right) = P\left(Z \geq \frac{\lambda-13}{14}\right)$$

$$= \frac{\mu-28}{35} = -\left(\frac{\lambda-13}{14}\right)$$

$$\Rightarrow 5\lambda + 2\mu - 121 = 0 \text{ - - - - - - - - - - - - - - - - - - (4)}$$

Solving (3) and (4) we get, $\lambda = 7.57$ and $= 41.571$

**EXAMINATION QUESTIONS :**

**( 2016)**

(ii) Show that if $X$ is a random variable having a binomial distribution with the parameters $n$ and $\theta$, then the moment generating function of

$$Z = \frac{X - n\theta}{\sqrt{n\theta(1-\theta)}}$$

approaches that of the standard normal distribution when $n \to \infty$.

## Uniform Distribution (Continuous)

A continuous random variable X is said to have a uniform distribution if its pdf is given by,

$$f(x) = \frac{1}{b-a}, a \leq xb$$

$$= 0 \text{ , elsewhere}$$

**Properties :**

1. a and b (a¡b) are the two parameters of the uniform distribution on (a,b).
2. This distribution is also known as rectangular distribution, since the curve *y= f(x)* describes a rectangle over the x-axis and between the ordinates at x=a and x=b.
3. The d.f., *f(x)* is given by

$$f(x) \begin{cases} 0, \text{ if } -\infty- < x < a \\ \frac{x-a}{b-a}, a < x < b \\ 1, b < x < \infty \end{cases}$$

### Moments:

Mean = E($x$)

$$= \int_a^b x f(x) dx$$

$$= \int_a^b x \frac{1}{b-a} dx$$

$$= \frac{1}{b-a} \left( \frac{x^2}{2} \right)_a^b$$

$$= \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2}$$

Variance

$$V(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \int_a^b x^2 f(x)dx = \int_a^b x^2 \frac{1}{b-a}dx$$

$$\frac{1}{b-a}\left(\frac{x^3}{3}\right)_a^b = \frac{b^3-a^3}{3(b-a)}$$

$$= \frac{b^2+ab+a^3}{3}$$

Therefore,

$$V(X) = \frac{b^2+ab+a^2}{3} - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{4b^2+4ab+4a^2-3b^2-6ab-3a^2}{12}$$

$$= \frac{(b-a)^2}{12}$$

Also,

$$SD(X) = \frac{b-a}{\sqrt{12}}$$

**EXAMPLE.**

For a rectangular distribution,

$$f(x) = \frac{1}{2a}, \ -a < x < a, \text{ Show that } \mu_{2r} = \frac{a^{2r}}{2r+1}$$

**Solution.**

We have $E(X) = \int_{-a}^{a} x f(x) dx = \int_{-a}^{a} x \frac{1}{2a} dx = 0$

There fore,

$$\mu_{2r} = E[X - E(X)]^{2r} = E[X62r]$$

$$= \int_{-a}^{a} x^{2r} \frac{1}{2a} dx$$

$$= \frac{1}{2a}\left(\frac{x^{2r+1}}{2r+1}\right)_{-a}^{a}$$

$$= \frac{1}{2a} \frac{(a^{2r+} + a^{2r+1}}{2r+1} = \frac{a^{2r+1}}{2r+1}$$

## Exponential Distribution

Let X be a continuous r.v with pdf,

$$f(x) = \lambda e^{-\lambda x}, x > 0, \lambda > 0$$

Then X is defined to have an exponential distribution.

**OR**

$$f(x) = \begin{cases} \frac{1}{\beta} e^{-x/\beta} & \text{for } x > 0, \ \beta > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Where $\lambda = 1/\beta$.

**Moments:**

Mean,

$$E(X) = \int_0^\infty x f(x) dx$$

$$= \int_0^\infty x\lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty e - \lambda x x^{2-1} dx$$

$$= \lambda \frac{\Gamma 3}{\lambda^3} = \frac{2}{\lambda^2}$$

Variance,

$$V(X) = E(X^2) - [E(X)]^2$$

$$E(X2) = \int_0^\infty x^2 \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty e^{-\lambda x} x^{3-1} dx$$

$$= \lambda \frac{\Gamma 3}{\lambda^3} = \frac{2}{\lambda^2}$$

There fore,

$$V(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

**Moment Generating Function:**

$$M_x(t) = E(e^{tx})$$

$$= \int_0^\infty e^{tx} x f(x) dx = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty e^{-(\lambda - t)x} dx = \lambda \left[ \frac{e^{-(\lambda - t)x}}{-(\lambda - t)} \right]_0^\infty$$

$$= \frac{\lambda}{\lambda - t} - (1 - \frac{t}{\lambda})^{-1}$$

**EXAMPLE.**

If X1, X2, …., $X_n$ are $n$ independent random variables following exponential parameter λ, find the distribution of $y = \sum_{i=1}^{n}$

**Solution.**

Given that X~*exponential with parameter* λ

Therefore,

$$M_x(t) = (1-\frac{t}{\lambda})^{-1}$$

Then,

$$M_y(t) = M\sum x^i(t) = \prod_{i=1}^{n} M_{Xi}(t)$$

$$= \prod_{i=1}^{n}(1-\frac{t}{\lambda})^{-1}$$

$$= (1-\frac{t}{\lambda})^{-n}$$

This is the mgf of a gamma distribution with parameter n and λ. There fore the pdf of Y is given by,

$$f(y) = \frac{\lambda^n}{\Gamma} n e^{-\lambda}(y^{n-1}),\ y \geq 0$$

$$= 0, \text{ elsewhere.}$$