Chapter

15

Uniform Convergence of Sequences and series of Functions

15.1 INTRODUCTION

In this chapter we shall consider sequences whose members are real valued functions defined in a set S and distinguish between two types of convergence of such sequences.

Let f_n be a real valued function defined in a set S for each n.

Point-wise convergence. **(Kanpur 2010)**

To each $c \in S$, there corresponds a real number sequence $\{f_n(c)\}$ with values

$$f_1(c), f_2(c), ..., f_n(c) ...$$

We suppose that this sequence is convergent. In fact, we suppose that each of the sequences arising for different members of S is convergent. Thus we define in a natural way a real valued function say f, with domain S such that its value $f(c)$ for $c \in S$ is $\lim \{f_n(c)\}$.

The function f, thus defined, is referred to as the *Pointwise* limit of the sequence $\{f_n\}$ of functions. Also in this case, we say that the sequence is point-wise convergent.

Thus if a function f is the point-wise limit of the point-wise convergent sequence $\{f_n\}$ of functions defined in S, to each $c \in S$ and to each $\varepsilon > 0$ there corresponds an integer m such that

$$\forall\, n \geq m, \quad |f_n(c) - f(c)| < \varepsilon.$$

Of course, if we fix, the choice of m may depend upon the choice of c.

Uniform convergence. **[Kanpur 2010; Chennai 2011; Meerut 2005, 13]**

We say that a sequence $\{f_n\}$ of real valued functions with domain S is uniformly convergent, if there exists a real valued function ϕ, with domain S and to each $\varepsilon > 0$ there corresponds an integer m such that

$$\forall\, x \in S \quad \text{and} \quad \forall\, n \geq m. \quad |f_n(x) - \phi(x)| < \varepsilon.$$

Also in this case we say that the function ϕ is the uniform limit of the sequence $\{f_n\}$.

It may be easily seen that

Uniform convergence $\Rightarrow$ *point-wise convergence*

and that in the event of uniform convergence

Uniform limit = Point-wise limit.

As a result, we shall denote the uniform limit of $\{f_n\}$ by f instead of ϕ.

It should, however, be remembered that every point-wise convergent sequence is not uniformly convergent as is illustrated by the following counter-exmaple.

Let $$f_n(x) = \frac{nx}{1 + n^2 x^2}, x \in R.$$

Here $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+n^2x^2} = \lim_{n\to\infty} \frac{x}{nx^2+1/n} = 0, \ \forall x \in R,$

showing that the sequence $\{f_n\}$, is point-wise convergent with point-wise limit f such that

$$f(x) = 0 \qquad \forall x \in R.$$

We shall now show that the convergence is *not* uniform in any interval $[a, b]$ with, 0, as an interior point. *(Agra 1998, 2004, Delhi B.Sc. Maths (H) 2002)*

Suppose that $\{f_n\}$ is uniformly convergent in $[a, b]$ so that the point-wise limit f is also the uniform limit.

Let $\varepsilon > 0$ be given. Then there exists m such that $\forall x \in [a,b]$ and $\forall n \geq m$.

$$\left|\frac{nx}{1+n^2x^2} - 0\right| < \varepsilon.$$

We take $\varepsilon = 1/4$. Now there exists an integer k such that $k \geq m$ and $1/k \in [a,b]$.

Taking $n = k$ and $x = 1/k$, we have $(nx)/(1+x^2x^2) < 1/2$,

which is not less that 1/4.

Thus we arrive at a contradiction and as such see that the sequence is *not* uniformly convergent in any interval $[a, b]$ with, 0, as an interior point even though it is point-wise convergent there.

Exercise. Show that the sequence $<f_n>$ where $f_n(x) = (nx)/(1+n^2x^2)$ does not converge uniformly on **R**. **(Meerut 2010)**

15.2 CAUCHY'S GENERAL PRINCIPLE OF UNIFORM CONVERGENCE

(Necessary and sufficient condition for uniform convergence) **(Meerut 2011)**

Theorem. *A necessary and sufficient condition for a sequence $\{f_n\}$ of functions defined in a set S to be uniformly convergent is that to each $\varepsilon > 0$, there corresponds m such that $\forall n \geq m, \forall p \geq 1$ and $\forall x \in S$,* $\left|f_{n+p}(x) - f_n(x)\right| < \varepsilon.$ **[Himachal 2013]**

Purvanchal 2006; Kanpur 2005, 13; Kanpur 2013; Delhi Maths (H) 1999, 2001, 2002, 06]

Proof. The condition is necessary. Let the sequence $\{f_n\}$ be uniformly convergent with f as its uniform limit.

Let $\varepsilon > 0$ be given. Then there exists m such that $\forall n \geq m$ and $\forall x \in S$,

$$\left|f_n(x) - f(x)\right| < \varepsilon/2. \quad \text{...(1)}$$

Also, since $\forall n \geq m, \ \forall p \geq 1$ and $\forall x \in S$,

$$\left|f_{n+p}(x) - f(x)\right| < \varepsilon/2, \quad \text{...(2)}$$

it follows that $\forall n \geq m, \ \forall p \geq 1$ and $\forall x \in S$

$$\left|f_{n+p}(x) - f_n(x)\right| = \left|f_{n+p}(x) - f(x) + f(x) - f_n(x)\right|$$

$$\leq \left|f_{n+p}(x) - f(x)\right| + \left|f(x) - f_n(x)\right|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon, \text{ using (1) and (2).}$$

The condition is sufficient. From Cauchy's principle of convergence as proved in theorem III of Art. 5.9 of chapter 5 it follows that the sequence is pointwise convergent. All that we have now to show is that the convergence is uniform. Let f be the point-wise limit of the sequence $\{f_n\}$. Let $\varepsilon > 0$ be given. Then there exists m such that $\forall\, n \geq m, \forall\, p \geq 1, \;\forall\, x \in S$,

$$\left|f_{n+p}(x) - f_n(x)\right| < \frac{1}{2}\varepsilon \quad \Rightarrow \quad f_n(x) - \frac{1}{2}\varepsilon < f_{n+p}(x) < f_n(x) + \frac{1}{2}\varepsilon.$$

Keeping n fixed and letting p tent to infinity, we see that $\forall\, n \geq m$ and $\forall\, x \in S$

$$f_n(x) - \varepsilon/2 \leq f(x) \leq f_n(x) + \varepsilon/2 \quad \Rightarrow \quad \left|f_n(x) - f(x)\right| \leq \varepsilon/2 < \varepsilon.$$

Thus the convergence is uniform.

EXAMPLES

Ex. 1. *Show that the sequence* $\{f_n\}$ *of functions where* $f_n(x) = n/(x+n)$, *is uniformly convergent in* [0, *k*] *whatever k may be, but not uniformly convergent in* $[0, \infty[$.

(Himachal 2013; Delhi B.A. (Prog.) III, 2014, 15; Delhi B.Sc. (Prog.) II, 2015)

Sol. The sequence $\{f_n\}$ is point-wise convergent $\forall\, x \geq 0$ and the point-wise limit f is given by

$$f(x) = 1 \;\; \forall\, x \geq 0$$

Let $\varepsilon > 0$ be given. We have

$$\left|f_n(x) - f(x)\right| = \frac{x}{x+n} < \varepsilon \quad \text{if} \quad n > x\left(\frac{1}{\varepsilon} - 1\right).$$

Let $m(\varepsilon, x)$ denote the integer just greater than $x(1/\varepsilon - 1)$. Obviously $m(\varepsilon, x)$ increases as x increases and $\to \infty$ as $x \to \infty$ so that it is not possible to choose any number m such that $\forall\, n \geq m$ and $\forall\, x \geq 0$,

$$\left|f_n(x) - f(x)\right| < \varepsilon$$

so that the convergence is *not* uniform in $[0, \infty[$.

Now consider the interval [0, *k*].

Let m be a integer greater than $k(1/\varepsilon - 1)$. We then see that $\forall\, n \geq m$ and $\forall\; x \in [0, k]$,

$$\left|f_n(x) - f(x)\right| < \varepsilon$$

so that the convergence is uniform in [0, *k*].

Ex. 2. *Show that the sequence* $\{f_n\}$ *where* $f_n(x) = x^n$ *is uniformly convergent in* [0, *k*] *where k is a number less than* 1 *and only point-wise convergent in* [0, 1]. **(Himachal 2013)**

[Delhi B.Sc. (Prog) 2008; Agra 2003; Delhi B.A. (Prog.) III 2014; Purvanchal 2006 Bhopal 2000; Rewa 2001; Ravishankar 2001; Agra 2003, Delhi B.Sc. Physics (H) 2004]

Sol. The given sequence is point-wise convergent in [0, 1] and the point-wise limit f is given by

$$f(x) = \begin{cases} 0, & \text{when } 0 \leq x < 1 \\ 1, & \text{when } x = 1 \end{cases}$$

Let $\varepsilon > 0$ be given. Then for $0 < x \leq k < 1$, we have

$$\left|f_n(x) - f(x)\right| = x^n < \varepsilon$$

$$\Rightarrow \quad (1/x)^n > (1/\varepsilon) \quad \Rightarrow \quad n \log(1/x) > \log(1/\varepsilon), x \neq 0 \quad \Rightarrow \quad n > (\log 1/\varepsilon) \div \log(1/x).$$

$\forall n \geq 1$.

Also if $x = 0$, $|f_n(x) - f(x)| = 0 < \varepsilon$ $\forall n \geq 1$.

Let $m(\varepsilon, x)$ denote the integer next greater than 1 and $\log(1/\varepsilon) \div \log(1/x)$.

Now, $m(\varepsilon, x)$ increases and $\to \infty$ as $x \to 1$, so that there does not exist an m such that

$$|f_n(x) - f(x)| < \varepsilon$$

$\forall n \geq m$ and $\forall x \in [0,1]$,

so that the convergence is *not* uniform in [0, 1].

Now suppose that k is a number such that $0 \leq k < 1$.

We see that in $[0, k]$, the greatest value of $\log(1/\varepsilon) \div \log(1/x)$ is $((\log(1/\varepsilon)) \div (\log 1/k))$.

Let any integer greater than this value be denoted by m. Then we see that $\forall n \geq m$ and

$$|f_n(x) - f(x)| < \varepsilon$$

$\forall x \in [0, k]$,

so that the convergence is uniform in $[0, k]$.

Note : A point, like $x = 1$ which is such that the sequence is not uniformaly conversent in any interval containing $x = 1$, is known as *a point of non-uniform convergence.*

Ex. 3. *Show that if* $f_n(x) = nxe^{-nx^2}$, *the sequence* $\{f_n\}$ *is point-wise, but not uniformly convergent in* $[0, k[, k > 0$.

(Pune 2010; Delhi B.Sc. (Prog), III, 2010, 11; Meerut 2003, 04, 05, 11, 07, 11)

Sol. It may be easily seen that the sequence is point-wise convergent and that the point-wise limit is the function f such that $\forall x, f(x) = 0$.

If possible, let the sequence be uniformly convergent in $[0, k[$, so that, $\varepsilon > 0$ being given, there exists m such that $\forall n \geq m$ and $\forall x \geq 0$

$$|f_n(x) - f(x)| = nxe^{-nx^2} < \varepsilon. \quad \text{...(I)}$$

Let m_0 be an integer greater than m and $e^2\varepsilon^2$ and let $x = 1/m_0$. Then the inequality (I) holds for $x = 1/\sqrt{m_0}$ and $n = m_0$. These give $\sqrt{m_0}/e < \varepsilon \Leftrightarrow m_0 < e^2\varepsilon^2$ so that we arrive at a contradiction.

Thus the convergence is *not* uniform in $[0, \infty[$, 0 being *a* point of non-uniform convergence of $< f_n(x) >$.

EXERCISES

1. Show that the sequence f_n where $f_n(x) = e^{-nx}$ is point-wise but not uniformly convergent in $[0, \infty]$. Also show that the convergence is uniform in $[k, \infty[$; k being any positive number. *(Agra 2000)*
2. Show that the sequence $\{e^{-nx}\}$ is uniformaly convergent in any interval $[a, b]$, where a and b are positive numbers but only point-wise in $[a, b]$.
3. Show that the sequence $\{f_n(x)\}$, where $f_n(x) = \tan^{-1} nx, x \geq 0$ is uniformaly convergent in any interval $[a, b]$, $a > 0$ but is only pointwise convergent in $[0, b]$ *(Himanchal 2008; Delhi Maths (H) 2000, 09; Jiwaji 2000)*
4. Show that the sequence $< f_n >$ defined as $f_n(x) = x^n/n$ on (*i*) $]-\infty, \infty[$ is not uniformly convergent. (*ii*) [0, 1] converges uniformly to 0
5. Show that the sequence $< f_n >$, where $f_n(x) = (n^2 x) / (1+n^2 x^2)$ is not uniformly convergent on [0, 1].

6. Show that the sequence $< f_n >$, where $f_n(x) = (nx) / (nx + 1)$ is uniformly convergent on $[a, b]$, $a > 0$ but in only pointwise convergent on [0, b].

7. Show that $< x_n >$ where $f_n(x) = (n^2x)/1 + n^4x^2)$ is not uniformly convergent on [0, 1]

15.3 A TEST FOR UNIFORM CONVERGENCE OF SEQUENCE OF FUNCTIONS

In order to test whether a given sequence $< f_n(x)$ in uniformly convergent or not in a given interval, so far we have been using the definition of uniform convergence. Accordingly, we tried to get $m \in N$, independent of x, which is not easy in practice. This method can be replaced by an easy method given in the following theorem.

Theorem (M_n-test). *Let $< f_n >$ be a sequence of functions defined on an interval I such that* $\lim_{n\to\infty} f_n(x) = f(x) \forall x \in [a, b]$ *and let* $M_n = \sup \{ | f_n(x) - f(x) | : x \in [a, b]\}$.

Then $< f_n >$ converges uniformly on $[a, b]$ if and only if $M_n \to 0$ as $n \to \infty$.

[Delhi Maths (Prog) 2008; Kanpur 2005, 06, 10; Bhopal 2004; Himachal 2013, 2014, Delhi B.Sc. (Prog.) III 2008, 2012; Meerut 2012]

Proof. The condition is necessary. Let $< f_n >$ converge uniformly to f on $[a, b]$. Then, for a given $\varepsilon > 0$, there exists a positive integer m such that

$$| f_n(x) - f(x) | < \varepsilon \ \forall\, x \geq m \text{ and } \forall\, x \in [a, b]$$

$$\Rightarrow \quad \sup \{| f_n(x) - f(x) | : x \in [a, b]\} < \varepsilon \ \forall\ n \geq m$$

$$\Rightarrow \quad M_n < \varepsilon \ \forall\ n \geq m \quad \Rightarrow \quad |M_n - 0| < \varepsilon \ \forall\ n \geq m$$

$$\Rightarrow \quad M_n \to 0, \text{ as } n \to \infty$$

The condition is sufficient. Let $M_n \to 0$ and $n \to \infty$. Then for a given $\varepsilon > 0$, there exists a positive integer m such that

$$| M_n - 0 | < \varepsilon \ \forall\, n \geq m \quad \Rightarrow \quad M_n < \varepsilon \ \forall\, n \geq m$$

$$\Rightarrow \quad \sup \{| f_n(x) - f(x) | : x \in [a, b]\} < \varepsilon \ \forall\, n \geq m$$

$$\Rightarrow \quad |f_n(x) - f(x) | < \varepsilon \ \forall\, n \geq m, \forall\, x \in [a, b]$$

$$\Rightarrow \quad < f_n > \text{ converges uniformly to } f \text{ on } [a, b]$$

EXAMPLES

Ex. 1. *Prove that the sequence $< f_n >$, where $f_n(x) = x / (1 + nx^2)$ converges uniformly on any closed intervals I.* **[Bhopal 1998; Indore 2000; Meerut 2001, 04, 08, 13; Delhi Maths (H) 2002, 03; 06, 07; Kanpur 2003, 2007; Delhi B.Sc. Physics (H) 2000]**

Sol. Here pointwise limit $= f(x) = \lim_{n\to\infty} f_n(x) = 0, \ \forall\, x \in \text{I}$

$$\therefore \quad | f_n(x) - f(x) | = \left| \frac{x}{1+nx^2} - 0 \right| = \left| \frac{x}{1+nx^2} \right| = | y |, \text{ say} \qquad \text{...(1)}$$

where

$$y = x / (1+nx^2) \qquad \text{...(2)}$$

From (2),

$$\frac{dy}{dx} = \frac{(1+nx^2)\cdot 1 - x \cdot 2nx}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2} \qquad \text{...(3)}$$

For maximum and minimum value of y, we have

$$dy/dx = 0 \quad \Rightarrow \quad 1 - nx^2 = 0 \quad \Rightarrow \quad x = 1/\sqrt{n} \in \text{I}$$

From (3),

$$\frac{d^2 y}{dx^2} = \frac{(1+nx^2)^2 \cdot (-2nx) - (1-nx^2)\cdot 2(1+nx^2)\cdot 2nx}{(1+nx^2)^4}$$

or $$\frac{d^2y}{dx^2} = \frac{-2nx\,(1+nx^2) - 4nx\,(1-nx^2)}{(1+nx^2)^3}$$

When $x = 1/\sqrt{n}$, $d^2y/dx^2 = -(\sqrt{n}/2) < 0$,

showing that y is maximum when $x = 1/\sqrt{n}$ and from (2) the maximum value of $y = 1/2\sqrt{n}$

$\therefore$ $M_n = \sup\{\,|f_n(x) - f(x)| : x \in I\,\} = \sup\{|y| : x \in I\} = 1/2\sqrt{n}$

Since $M_n \to 0$ as $n \to \infty$, $<f_n>$ is uniformly convergent on any closed interval I.

Ex. 2. *Show that the sequence* $<f_n>$, *where* $f_n(x) = nx\,(1-x)^n$ *is not uniformly convergent on* [0, 1].(*Pune 2010; Rajasthan 2010; Bhopal 2003; Kanpur 2004; Meerut 08, 09; Agra 2000,*

Sol. When $x = 0, f_n(x) = 0\ \forall\, n \in N$; when $x = 1, f_n(x) = 0\ \forall\, n \in N$

Hence $f(x) = \lim_{n\to\infty} f_n(x) = 0$ when $x = 0$ and $x = 1$.

Again, for $0 < x < 1$, we have

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nx(1-x)^n = \lim_{n\to\infty} \frac{nx}{(1-x)^{-n}} \qquad \left[\text{Form } \frac{\infty}{\infty}\right]$$

$$= \lim_{n\to\infty} \frac{x}{-(1-x)^{-n}\log(1-x)} = \lim_{n\to\infty} \frac{-x(1-x)^n}{\log(1-x)} \qquad \text{[using L' Hopital's rule]}$$

$= 0$ as $0 < x < 1$ so that $(1-x)^n \to 0$ as $n \to \infty$

Thus, we have $f(x) = 0\ \forall\, x \in [0, 1]$

$\therefore$ $|f_n(x) - f(x)| = |\,nx\,(1-x)^n - 0\,| = nx(1-x)^n = y$, say

where $y = nx\,(1-x)^n$

$\therefore$ $dy/dx = n(1-x)^n - n^2x(1-x)^n = n\,(1-x)^{n-1}\,\{1-(n+1)\,x\}$

For maximum and minimum value of y, we have

$dy/dx = 0 \quad \Rightarrow \quad x = 1/(n+1)$

From (3), $d^2y/dx^2 = -n(n-1)\,(1-x)^{n-2}\,\{1-(n+1)x\} - n\,(n+1)\,(1-x)^{n-1}$.

When $x = \dfrac{1}{n+1}$, $\dfrac{d^2y}{dx^2} = -n(n+1)\cdot\left(\dfrac{n}{n+1}\right)^{n-1} < 0$,

showing that y is maximum when $x = 1/(n+1)$ and from (2),

$$\text{the maximum value of } y = \frac{n}{n+1}\left(1-\frac{1}{n+1}\right) = \left(1-\frac{1}{n+1}\right)^{n+1}$$

$\therefore$ $M_n = \sup\{\,|f_n(x) - f(x)| : x \in [0, 1]\,\} = \sup\{\,|y| : x \in [0, 1]\,\} = \left(1-\dfrac{1}{n+1}\right)^{n+1}$, by (4)

$\therefore$ $M_n \to e^{-1}$ as $n \to \infty$. Since M_n does not tend to 0 as $n \to \infty$, the sequence $<f_n>$ is not uniformly anvergent on [0, 1]. Here 0 is a point of non-uniform conversence because $x = 1/(n+1) \to 0$ as $n \to \infty$.

Ex. 3. *Show that the sequence* $<f_n>$ *where* $f_n(x) = (\sin n\,x)/\sqrt{n}$ *is uniformly convergent on* $[0, \pi]$.

(Himachal 2014; Delhi B.A. (Prog) III 2011; Kanpur 2005; Agra 199

Sol. Here $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \dfrac{1}{\sqrt{n}}\cdot \sin n\,x = 0 \quad \forall\, x \in [0, \pi]$

$\therefore$ $$|f_n(x) - f(x)| = \left|\frac{\sin nx}{\sqrt{n}} - 0\right| = |y|, \text{ say} \qquad ...(1)$$

where $$y = \frac{\sin nx}{\sqrt{n}} \text{ so that } \frac{dy}{dx} = \sqrt{n} \cos nx \qquad ...(2)$$

For maximum and minimum value of y, we have

$dy/dx = 0 \Rightarrow \cos nx = 0 \Rightarrow nx = \pi/2$ or $x = \pi/2n \in [0, \pi]$

Again, $d^2y/dx^2 = -n^{3/2} \sin nx$

$\Rightarrow$ When $x = \pi/2n$, $d^2y/dx^2 = -n^{3/2} \sin(\pi/2) = -n^{3/2} < 0$,

showing that y is maximum when $x = \pi/2n$ and

the maximum value of $y = (1/\sqrt{n}) \times \sin(\pi/2) = 1/\sqrt{n}$. ...(3)

Moreover $x = \pi/2n \to 0$ as $n \to \infty$

From (1), (2) and (3) $M_n = \sup \{|f_n(x) - f(x)| : x \in [0, \pi]\} = 1/\sqrt{n}$

Since $M_n \to 0$ and $n \to \infty$, $< f_n >$ is uniformly convergent on $[0, \pi]$.

EXERCISE

1. Show that the sequence $< f_n >$ where $f_n(x) = nx/(1+n^2x^2)$ is not uniformly convergent on any interval containing zero. *(G.N.D.U. Amristar 2002; Jabalpur 2000; Agra 2008, 10; Kanpur 2007, Himanchal 2014; Meerut 2000; 02, 05, 06; Delhi B.Sc. (Prog.) II, 2016)*
2. Show that the sequence $< f_n >$ where $f_n(x) = nx\, e^{-nx^2}$, is not uniformally convergent on (*i*) [0, 1] (*ii*) $[0, k]$, $k > 0$ *(Himanchal Pradesh 2002)*
3. Show that if $f_n(x) = n^2x/(1+x^4x^2)$, then $< f_n >$ converges non-uniformly on [0, 1].
4. Show that the sequence $< f_n >$, where $f_n(x) =$
   (*i*) $n^2x / (1+n^3x^2)$ is not uniformly convergent on [0, 1] ***(Himachal 2014)***
   (*ii*) $nx / (1 + n^3x^2)$ converges uniformly on any closed interval $[a, b]$
   (*iii*) $n^2x / (1 + n^2x^2)$ is not uniformly convergent on [0, 1]
   (*iv*) $(1/n) \times e^{-nx}$ converges uniformly to 0 on $[0, \infty[$ ***(Kanpur 2011)***
5. Show that the sequence $< f_n >$ is uniformly convergent for all $x \geq 0$, when
   (*i*) $f_n(x) = x / (n + x^2)^2$ (*ii*) $x/n(1 + nx^2)$
6. Show that the sequence $< x^{n-1}(1 - x) >$ is uniformaly convergent on [0, 1] *(Sagar 1995)*
7. If a sequence $< f_n(x) >$ conveges uniformly to $f(x)$ on $[a, b]$ and x_0 is a point of $[a, b]$ such that $\lim_{n \to x_0} f_n(x) = a_n$ $n = 1, 2, 3, ...,$ then prove that
   (*i*) $< a_n >$ converges (*ii*) $\lim_{x \to x_0} f(x) = \lim_{n \to \infty} a_n$. *(Delhi Maths (H) 2000, 2005)*

15.4 CONTINUITY OF THE UNIFORM LIMIT OF A UNIFORMLY CONVERGENT SEQUENCE OF CONTINUOUS FUNCTIONS. *[Delhi B.Sc. (Prog.) II, 2016]*

Theorem. *If $< f_n >$ is a sequence of continuous functions on an interval $[a, b]$ and if $f_n \to f$ uniformly on $[a, b]$, then f is continuous on $[a, b]$.* ***[Bilaspur 2000; Kanpur 2003, 04; Ravishankar 2000; G.N.D.U. Amritsar 2000, 02, Jiwaji 2001; Kurukshetra 2003; Himachal 2013, 2014, 2015; Meerut 2005; Delhi B.Sc. (Hons) II, 2011]***

Proof. Consider a uniformly convergent sequence $\{f_n\}$ of continuous functions defined in $[a, b]$. Let the function f be the limit which is, of course, the uniform limit of the sequence. It will now be proved that f is a continuous function. Let c be any point of $[a, b]$.

We have $\forall\ x$ and $\forall\ m \in N$,

$$|f(x) - f(c)| = |f(x) - f_m(x) + f_m(x) - f_m(c) + f_m(c) - f(c)|$$
$$\leq |f(x) - f_m(x)| + |f_m(x) - f_m(c)| + |f_m(c) - f(c)|. \qquad ...(1)$$

Let $\varepsilon > 0$ be given. As $\{f_n\}$ converges uniformly to f, there exists $m \in N$, such that $\forall\, x \in [a, b]$ and $\forall\, n \geq m$, we have

$$|f(x) - f_n(x)| < \varepsilon/3$$

In particular, we see that $\forall\, x \in [a,b]$,

$$|f(x) - f_m(x)| < \varepsilon/3 \quad \Rightarrow \quad |f(c) - f_m(c)| < \varepsilon/3 \qquad \text{... (2)}$$

As f_m is continuous at c, there exists $\delta > 0$ such that $\forall\, x \in]c-\delta, c+\delta[$

$$|f_m(x) - f_m(c)| < \varepsilon/3 \qquad \text{... (3)}$$

From (1), (2) and (3), we have

$$|f(x) - f(c)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \quad \text{or} \quad |f(x) - f(c)| < \varepsilon.$$

Thus we see that there exists $\delta > 0$ such that $\forall\, x \in]c-\delta, c+\delta[$

$$|f(x) - f(c)| < \varepsilon,$$

and as such f is continuous at c and, therefore, also at every point of the domain.

Note1. Uniform convergence of the sequence $< f_n >$ is only a sufficient condition but not a necessary for the continuity of the limit function f, i.e. if the limit function f is continuous on $[a, b]$, then it is not necessary that the sequence $< f_n >$ is uniformly convergent on $[a, b]$.

Note 2. From the above theorem, it follows that if the limit function f is discontinuous on $[a, b]$, then the sequence $< f_n >$ of continuous function cannot be uniformly convergent on $[a, b]$. Therefore, the above theorem provides us an easy method to prove that a certain sequence is not uniformly convergent.

Example 1. *Show that the sequence* $< f_n >$ *where* $f_n(x) = \tan^{-1} nx$ *is not uniformaly convergent on* [0, 1]. *(Delhi Maths (H) 1998, 2000)*

Solution. The limit function f is given by

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \tan^{-1} n\,x = \pi/2 \ \text{ for } 0 < x \leq 1$$

When $x = 0$, the sequence $< f_n >$ converges to 0

Thus,
$$f(x) = \begin{cases} \pi/2, \text{ if } 0 < x \leq 1 \\ 0, \text{ if } x = 0 \end{cases}$$

Clearly, f is discontinuous at $x = 0$ and so f is discontinuous on [0, 1].

Also, $f_n(x) = \tan^{-1} nx,\ 0 \leq x \leq 1$ is continous on [0, 1] $\forall\, n \in N$

Thus, $< f_n >$ is a sequence of continuous functions and its limit function f is discontinuous on [0, 1]. Hence the $< f_n >$ is not uniformly conversent on [0, 1].

Example 2. Let f_n be defined by $f_n(x) = nx/(1+x^2x^2)$, in any domain $[a, b]$ with 0 as an interior point. If has been seen on page 15.2 that the sequence is point wise convergent but not uniformly convergent. Also the point wise limit f is given by

$$f(x) = 0 \ \ \forall\, x \in [a, b]$$

Thus we see that while convergence is not uniform, the point wise limit itself is a continuous function.

Example 3. *Let* f_n *be defined by* $f_n(x) = 1 - |1 - x^2|^n$ *in the domain*

$$\{x : |1 - x^2| \leq 1\} = [-\sqrt{2}, \sqrt{2}].$$

The given sequence $\{f_n\}$ is point-wise convergent in $\left[-\sqrt{2}, \sqrt{2}\right]$ with the point-wise limit f, where

$$f(x) = \begin{cases} 1 \text{ when } \left|1-x^2\right| < 1 \\ 0 \text{ when } \left|1-x^2\right| = 1 \end{cases} \quad \text{i.e., for } x = 0, \pm\sqrt{2}.$$

Thus we see that the point-wise limit is not continuous for $x = 0$ even though each f_n is continuous there at.

Thus we may conclude that the sequence cannot be uniformly convergent in $\left[-\sqrt{2}, \sqrt{2}\right]$. We may also see this fact directly as follows :

Suppose that the sequence is uniformly convergent in $\left[-\sqrt{2}, \sqrt{2}\right]$ so that f is the uniform limit. Take $\varepsilon = 1/2$. Then there exists $m \in N$ such that

$$\left|f_n(x) - f(x)\right| < 1/2, \quad \forall\, n \geq m \quad \text{and} \quad \forall\, x \in \left[-\sqrt{2}, \sqrt{2}\right].$$

In particular $\quad \forall\, x \in \left[-\sqrt{2}, \sqrt{2}\right], \quad \left|f_m(x) - f(x)\right| < 1/2.$

Now, $$\left|f_m(x) - f(x)\right| = \begin{cases} \left|1-x^2\right|^m & \text{when } \left|1-x^2\right| < 1 \\ 0 & \text{when } \left|1-x^2\right| = 1 \end{cases}$$

Since $\lim\limits_{x \to 0} \left|1-x^2\right|^m = 1$, there exists a neighbourhood of 0 for every x of which $\left|1-x^2\right|^m$ belongs to a neighbourhood

$$\left]1-\frac{1}{4}, 1+\frac{1}{4}\right[= \left]\frac{3}{4}, \frac{5}{4}\right[$$

of 1, and is as such greater than 1/2.

Thus, we arrive at a contradiction and as such we see directly that the convergence is *not* uniform in $\left[-\sqrt{2}, \sqrt{2}\right]$.

EXERCISE

1. Show that the sequence $< f_n >$, where $f_n(x) = x^n$, $0 \leq x \leq 1$ is not uniformly convergent on [0, 1]. ***[Delhi B.Sc. (Hons) II 2011]***
2. Show that the sequence $< f_n >$, where $f_n(x) = 1/(1+nx)$, $0 \leq x \leq 1$ is not uniformly convergent on [0, 1] **[Kanpur 2013**

15.5 INTEGRABILITY OF THE UNIFORM LIMIT OF A UNIFORMLY CONVERGENT SEQUENCE OF INTEGRABLE FUNCTIONS

Theorem. Let $\{f_n\}$ be a uniformly convergent sequence with uniform limit f on $[a, b]$ and let f_n be integrable on $[a, b]$ $\forall\, n \in N$. Then the limit f is itself integrable and

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} \int_a^b f_n(x)\, dx. \quad \text{[Mumbai 2010; Meerut 2001, 03, 04]}$$

(Kumaun 1999, Himachal 2002, Agra 2001, Delhi Maths (H) 2001, 06, 07)

Proof. Let $\varepsilon > 0$ be given so that there exists $m \in N$ such that $\forall\, x \in [a,b]$ and $\forall\, n \geq m$,

$$\left|f_n(x) - f(x)\right| < \varepsilon / 4\,(b-a).$$

In particular, we have $\forall\, x \in [a,b]$, $\quad \left|f_m(x) - f(x)\right| < \varepsilon / 4\,(b-a).$

We write $\quad f(x) - f_m(x) = R_m(x),$

so that we have defined a new function R_m.

As the function f_m is integrable, there exists a partition, say P, of $[a, b]$ such that the oscillatory sum $w(P, f_m)$ for the function f_m corresponding to the partition P is $< \varepsilon / 2$.

Also since $\quad \left|R_m(x)\right| < \varepsilon / 4\,(b-a)\ \forall\, x \in [a,b],$

the oscillation of R_m in each sub-interval is $< \varepsilon / 2\,(b-a)$.

implying that $\quad w(P, R_m) < \dfrac{\varepsilon}{2\,(b-a)}\,(b-a) = \dfrac{\varepsilon}{2}.$

Also $\quad f = f_m + R_m \Rightarrow w(P, f) \leq w(P, f_m) + w(P, R_m).$

Thus if $\varepsilon > 0$ be given, there exists a partition P of $[a, b]$ such that $w(P, f) < \varepsilon$, and accordingly f is integrable.

Now, we prove that $\quad \displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx.$

Let $\varepsilon' > 0$ be given so that there exists $m' \in N$ such that $\forall\, x \in [a,b]$ and $\forall\, n \geq m'$

$$\left|f(x) - f_n(x)\right| < \varepsilon' / (b-a)$$

$$\Rightarrow \quad -\varepsilon'/(b-a) < f(x) - f_n(x) < \varepsilon'/(b-a).$$

$$\Rightarrow \quad -\varepsilon'/(b-a) < (f - f_n)(x) < \varepsilon'/(b-a).$$

Now f_n is integrable and f has also been proved integrable. Thus $f - f_n$ is integrable. We have

$$\left|\int_a^b (f - f_n)\ (x)\,dx\right| < \varepsilon'.$$

Thus if $\varepsilon' > 0$ is given, there exists $m' \in N$ such that $\forall\, n \geq m'$

$$\left|\int_a^b f(x)\,dx - \int_a^b f_n(x)\,dx\right| < \varepsilon' \Rightarrow \lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Cor. *It may be proved without difficulty that the sequence*

$$\left\{\int_a^t f_n(x)\,dx\right\}$$

of functions is uniformly convergent with limit

$$\int_0^t f(x)\,dx,\ t \in [a, b].$$

Note 1. The converse of the above theorem may not be true, *i.e.*, a sequence may converge to an integrable limit without being uniformly convergent.

15.6 DERIVABILITY OF THE POINT-WISE LIMIT OF A SEQUENCE OF DERIVABLE FUNCTIONS IF THE DERIVATIVES ARE CONTINUOUS AND THE SEQUENCE OF DERIVATIVES IS UNIFORMLY CONVERGENT

Theorem. *Let $\{f_n\}$ be a sequence of derivable functions with point-wise limit f. Let f_n' be continuous on $[a, b]$ $\forall n \in N$ and let the sequence $\{f_n'\}$ be uniformly convergent with ϕ as uniform limit on $[a, b]$. Then ϕ is derivable and derivative is equal to f, i.e.,*

$$\phi' = \left(\lim_{n\to\infty} \{f_n\}\right)' = \lim_{n\to\infty} \{f_n'\}$$

i.e., the derivative of the limit of the sequence is equal to the limit of the sequence of derivatives.

[G.N.D.U. Amritsar 2003; Jiwaji 2001; Ravishankar 2003]

Proof. As $\{f_n'\}$ is a uniformly convergent sequence of continuous functions, it follows that its uniform limit ϕ is a cootinuous function. It follows that

$$\int_a^t \phi(x)\,dx = \lim \int_a^t f_n'(x)\,dx = \lim \{f_n(t) - f_n(a)\}$$

$$= \lim\{f_n(t)\} - \lim\{f_n(a)\} = f(t) - f(a).$$

Also ϕ being continuous, we have,

$$\phi'(t) = f(t)\ \forall t \in [a, b] \Rightarrow \phi' = f.$$

Note. The above theorem provides a good negative test for the uniform convergence of $< f_n' >$. Accordingly, if the derivative of the limit of the sequence is not equal to the limit of the sequence of the derivates, then the sequence $< f_n'(x) >$ cannot be uniformly convergent.

Example. *Show that the sequence $< f_n >$ of functions where $f_n(x) = nx/(1+n^2x^2)$ converges to f where $f(x) = 0$ for all x and that the equation $f'(x) = \lim_{n\to\infty} f_n'(x)$ is true for all $x \neq 0$ but is false if $x = 0$. What you say about the uniform convergence of the sequence $< f_n' >$ in an interval containing zero* *(Delhi Maths (H) 2001)*

Sol. By example on page 15.1, the sequence $<f_n>$ converges pointwise to $f(x) = 0\ \forall x \in [0, 1]$

Now, $$f(x) = 0 \Rightarrow f'(x) = 0\ \forall x \in [0, 1] \quad ...(1)$$

For $x \neq 0$, $$f_n'(x) = \frac{n(1+n^2x^2) - 2n^2x \cdot nx}{(1+n^2x^2)^2} = \frac{n(1-n^2x^2)}{(1+n^2x^2)^2} = \frac{1/n^2 - x^2}{n(1/n^2 + x^2)}$$

$$\Rightarrow \lim_{n\to\infty} f_n'(x) = 0 \text{ for all } x \neq 0. \quad ...(2)$$

From (1) and (2), $$f'(x) = \lim_{n\to\infty} f_n'(x) \text{ is true for all } x \neq 0$$

Now, for $x = 0$, we have

$$f_n'(0) = \lim_{x\to 0} \frac{f_n(x) - f_n(0)}{x - 0} = \lim_{x\to 0} \frac{(nx)/(1+n^2x^2)}{x} = \lim_{x\to 0} \frac{n}{1+n^2x^2} = n$$

$$\Rightarrow f_n'(0) \to \infty \text{ as } n \to \infty$$

Thus at $x = 0$, $f'(x) = \lim_{n\to\infty} f_n'(x)$, is false.

Thus, $\left(\lim_{n\to\infty} < f_n >\right)' \neq \lim_{n\to\infty}\left\{f_n'\right\}$ in any interval containing zero. It follows, that $< f_n' >$ cannot converge uniformly in any interval containing zero.

EXERCISES

1. Show that the sequence $< f_n >$, where $f_n(x) = x/(1+nx^2)$ converges uniformly to a function f on [0, 1], and that the equation $f_n' = \lim_{n\to\infty} f_n'(x)$ is true if $x \neq 0$ and false if $x = 0$. Why so? *[Rohilkhand 1996, 97. Agra 1998]*
2. Show that the sequence $< f_n >$ of functions where $f_n(x) = nx/(1 + n^3x^2)$ converges uniformly to f on [0, 1] when $f(x) = 0 \ \forall\, x \in [0, 1]$. Check whether the equation $f'(x) = \lim_{n\to\infty} f_n'(x)$ holds on [0, 1]. *(Delhi Maths (H) 2002)*
3. (*a*) Show that the sequence $< f_n >$, where $f_n(x) = x - x^n/n$, converges uniformly on [0, 1]. Show that the sequence $< f_n' >$ of differentails does not converge uniformaly on [0, 1]. *(Delhi Maths (H) 1999)*

   (*b*) Decide whether or not the sequence $< f_n' >$ converges uniformly on [0, 1], where $f_n(x) = x - x^n/n$. *(Delhi B.Sc. Physics (H) 1999)*

15.7 INFINITE SERIES OF FUNCTIONS *(Himachal 2007; Kanpur 2007)*

We now consider series whose terms are functions defined in some set S. Let

$$\sum_{n=1}^{\infty} f_n(x)$$

be such a series. We write

$$S_n(x) = \sum_{n=1}^{\infty} f_n(x)$$

so that $\{S_n(x)\}$ is a sequence of functions.

We say that the series is point-wise convergent if the sequence $\{S_n(x)\}$ is point-wise convergent. Also the series is said to be uniformly convergent, if the sequence $\{S_n(x)\}$ is uniformly convergent. Also, the point-wise limit or the uniform limit of $\{S_n(x)\}$ as the case may be is said to be the point-wise sum or the uniform sum of the series and is denoted by $S(x)$. $S(x)$ is also known as sum function or limit function. Thus,

$$S(x) = \lim_{n\to\infty} S_n(x) = \sum_{n=1}^{\infty} f_n(x)$$

15.8 TEST FOR THE UNIFORM CONVERGENCE OF A SERIES

15.8.1 Cauchy's general principle of uniform convergence for series.

The necessary and sufficient condition for the uniform convergence in $[a, b]$ of a series Σf_n is that to every positive number, ε, there corresponds a positive integer m such that $\forall\, n \geq m$; $\forall\, p \geq 1$ and $\forall\, x \in [a,b]$,

$$\left|f_{n+1}(x) + f_{n+2}(x) + \dots + f_{n+p}(x)\right| < \varepsilon. \qquad \textbf{(Himachal 2014)}$$

This result is an immediate consequence of the corresponding result for sequences proved in Art. 15.2.

15.8.2. Weierstrass's M-test for Uniform Convergence

(Himanchal 2007, 08, 10) (Meerut 2006, 09, 13; Kanpur 2007, 09; Agra 2007, 09, 10, 12; Delhi B.Sc. (Hons) II 2011, Delhi B.Sc. (Prog.) II, 2015; Delhi B.A. (Prog.) III 2012, 2013)

Theorem. A series Σf_n will converge uniformly in $[a, b]$, if there exists a convergent series ΣM_n of positive numbers such that $\forall\ x \in [a,b]$,

$$|f_n(x)| \le M_n$$ *(Kanpur 2009, 10; Meerut 2010)*

Proof. Let ε be a positive number. Since ΣM_n is convergent, there exists a positive integer m such that

$$\left|M_{n+1} + M_{n+1} + ... + M_{n+p}\right| < \varepsilon,\ \forall\, n \ge m \text{ and } \forall\, p \ge 1. \quad ...(i)$$

Also, given that $$|f_n(x)| \le M_n\ \forall\, x \in [a, b] \quad ...(ii)$$

From (*i*) and (*ii*), we see that

$$\forall\, x \in [a, b], \forall\, n \ge m \text{ and } \forall\, p \ge 1,$$

$$\left|f_{n+1}(x) + f_{n+2}(x) + ... + f_{n+p}(x)\right| \le \left[M_{n+1} + M_{n+2} + ... + M_{n+p}\right] < \varepsilon.$$

Hence Σf_n is uniformly convergent in [*a*, *b*]

Remark. Note carefully that M_n must be independent of x for every $n = 1, 2, 3, ...$

EXAMPLES

Ex. 1. *Show that the following series are uniformly convergent for real values of* x :

(*i*) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin nx}{n^p}$, $p > 1$ *(Himanchal 2008, 09); (Kanpur 2012)* $\displaystyle\sum_{n=1}^{\infty} \frac{\cos nx}{n^4}, x \in R$ *(Delhi B.Sc. (Prog) II 2008)*

(*ii*) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ *(Kanpur 2005; Meerut 2009) (Delhi B.A. (Prog.) III 2012)* (*iii*) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ *(Delhi Maths (H) 2001)*

(*iv*) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin (x^2 + nx^2)}{n(n+1)}$ *(Himanchal 2004, 05) (Delhi Maths (H) 2001)*

Sol. (*i*) Here, $f_n(x) = (\sin nx) / n^p$ and so

$$|f_n(x)| = \left|\frac{\sin nx}{n^p}\right| = \frac{|\sin nx|}{n^p} \le \frac{1}{n^p} = M_n, \text{ say } \forall\, x \in R$$

Since $\displaystyle\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for p > 1, so by Weierstrass's M test, the given series converges uniformly for all real values of x.

(*ii*) and (iii). Left as exercises for the reader.

(*iv*) Here $f_n(x) = \{\sin (x^2+nx^2)\}/n(n+1)$

$$\therefore\ |f_n(x)| = \left|\frac{\sin(x^2 + nx^2)}{n(n+1)}\right| = \frac{|\sin(x^2 + nx^2)|}{n^2(1 + 1/n^2)} \le \frac{1}{n^2} = M_n \text{ for } \forall\, x \in R$$

Since $\displaystyle\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, so by Weierstrass's M-test, the given series is uniformly convergent for all real values of x.