2.2 Divergence and Curl of Electrostatic Fields

2.2.1 Field Lines, Flux, and Gauss's Law

In principle, we are *done* with the subject of electrostatics. Equation 2.8 tells us how to compute the field of a charge distribution, and Eq. 2.3 tells us what the force on a charge Q placed in this field will be. Unfortunately, as you may have discovered in working Prob. 2.7, the integrals involved in computing $\mathbf{E}$ can be formidable, even for reasonably simple charge distributions. Much of the rest of electrostatics is devoted to assembling a bag of tools and tricks for *avoiding* these integrals. It all begins with the divergence and curl of $\mathbf{E}$. I shall calculate the divergence of $\mathbf{E}$ directly from Eq. 2.8, in Sect. 2.2.2, but first I want to show you a more qualitative, and perhaps more illuminating, intuitive approach.

Let's begin with the simplest possible case: a single point charge q, situated at the origin:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}. \tag{2.10}$$

To get a "feel" for this field, I might sketch a few representative vectors, as in Fig. 2.12a. Because the field falls off like $1/r^2$, the vectors get shorter as you go farther away from the origin; they always point radially outward. But there is a nicer way to represent this field, and that's to connect up the arrows, to form **field lines** (Fig. 2.12b). You might think that I have thereby thrown away information about the *strength* of the field, which was contained in the length of the arrows. But actually I have not. The magnitude of the field is indicated by the *density* of the field lines: it's strong near the center where the field lines are close together, and weak farther out, where they are relatively far apart.

In truth, the field-line diagram is deceptive, when I draw it on a two-dimensional surface, for the density of lines passing through a circle of radius r is the total number divided by the circumference $(n/2\pi r)$, which goes like $(1/r)$, not $(1/r^2)$. But if you imagine the model in three dimensions (a pincushion with needles sticking out in all directions), then the density of lines is the total number divided by the area of the sphere $(n/4\pi r^2)$, which *does* go like $(1/r^2)$.

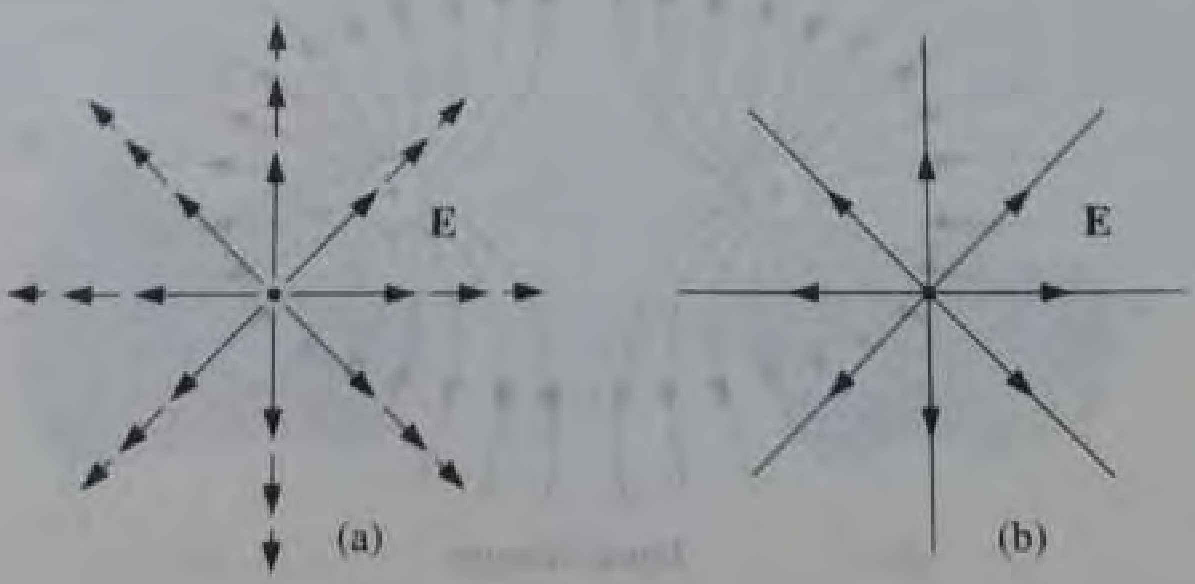
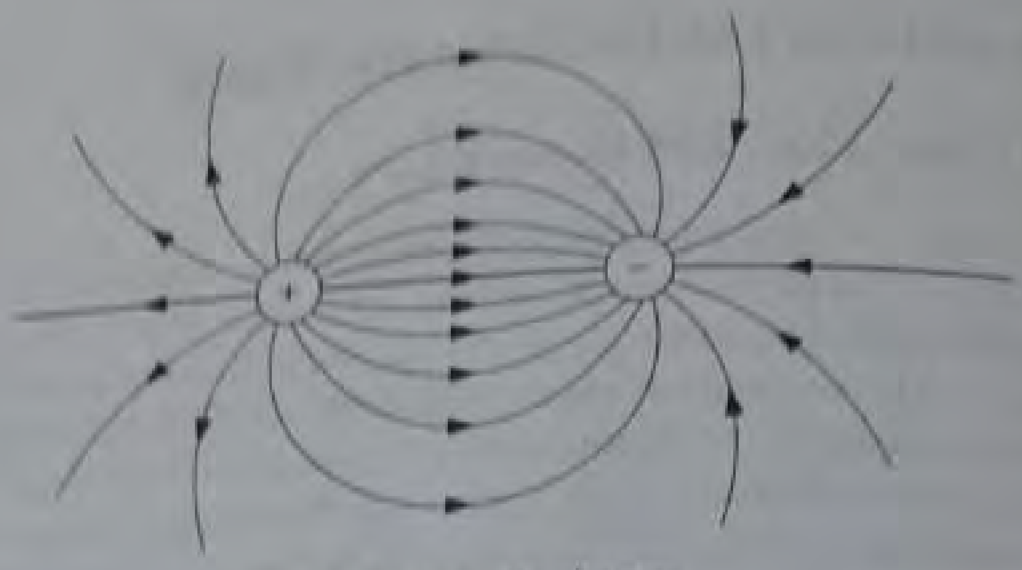


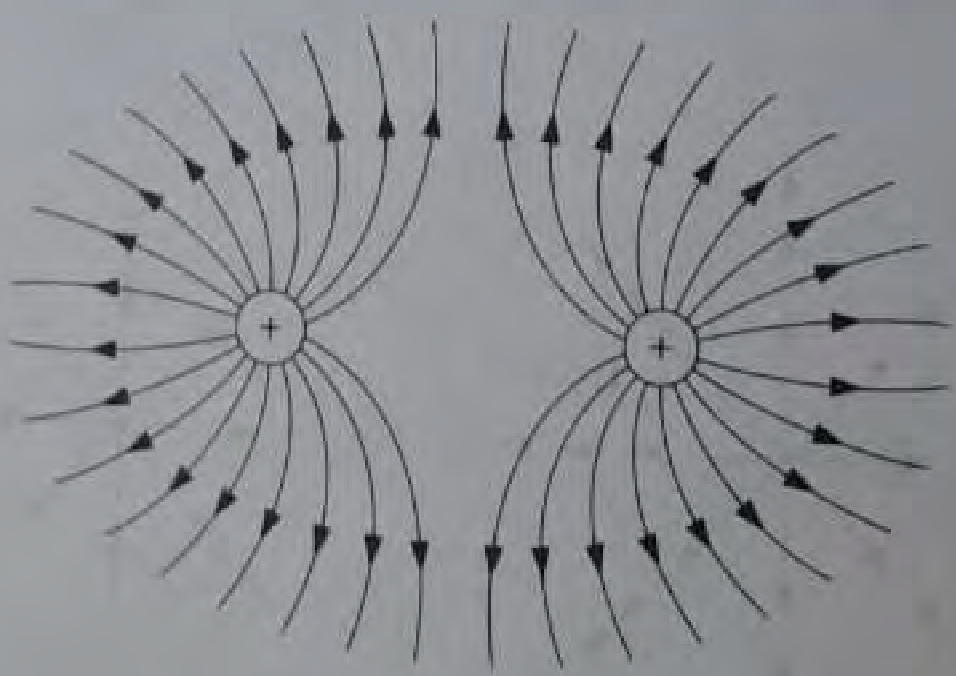
Figure 2.12



Equal but opposite charges

Figure 2.13

Such diagrams are also convenient for representing more complicated fields. Of course, the number of lines you draw depends on how energetic you are (and how sharp your pencil is), though you ought to include enough to get an accurate sense of the field, and you must be consistent: If charge q gets 8 lines, then $2q$ deserves 16. And you must space them fairly—they emanate from a point charge symmetrically in all directions. Field lines begin on positive charges and end on negative ones; they cannot simply terminate in midair, though they may extend out to infinity. Moreover, field lines can never cross—at the intersection, the field would have two different directions at once! With all this in mind, it is easy to sketch the field of any simple configuration of point charges: Begin by drawing the lines in the neighborhood of each charge, and then connect them up or extend them to infinity (Figs. 2.13 and 2.14).



Equal charges

Figure 2.14

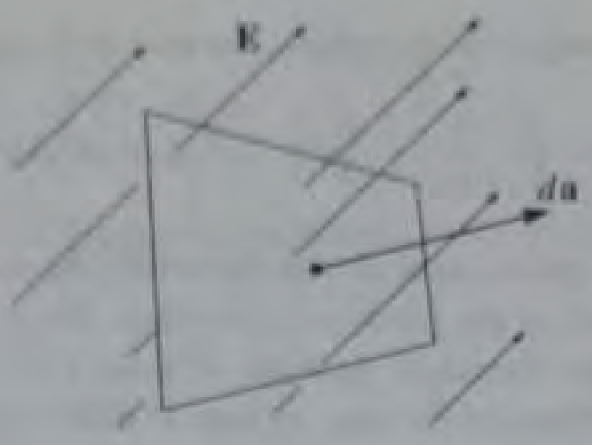


Figure 2.15

In this model the *flux* of $\mathbf{E}$ through a surface $\mathcal{S}$,

$$\Phi_E \equiv \int_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a}, \tag{2.11}$$

is a measure of the "number of field lines" passing through $\mathcal{S}$. I put this in quotes because of course we can only draw a representative *sample* of the field lines—the *total* number would be infinite. But *for a given sampling rate* the flux is *proportional* to the number of lines drawn, because the field strength, remember, is proportional to the density of field lines (the number per unit area), and hence $\mathbf{E} \cdot d\mathbf{a}$ is proportional to the number of lines passing through the infinitesimal area $d\mathbf{a}$. (The dot product picks out the component of $d\mathbf{a}$ along the direction of $\mathbf{E}$, as indicated in Fig. 2.15. It is only the area *in the plane perpendicular to* $\mathbf{E}$ that we have in mind when we say that the density of field lines is the number per unit area.)

This suggests that the flux through any *closed* surface is a measure of the total charge inside. For the field lines that originate on a positive charge must either pass out through the surface or else terminate on a negative charge inside (Fig. 2.16a). On the other hand, a charge *outside* the surface will contribute nothing to the total flux, since its field lines pass in one side and out the other (Fig. 2.16b). This is the *essence* of **Gauss's law**. Now let's make it quantitative.

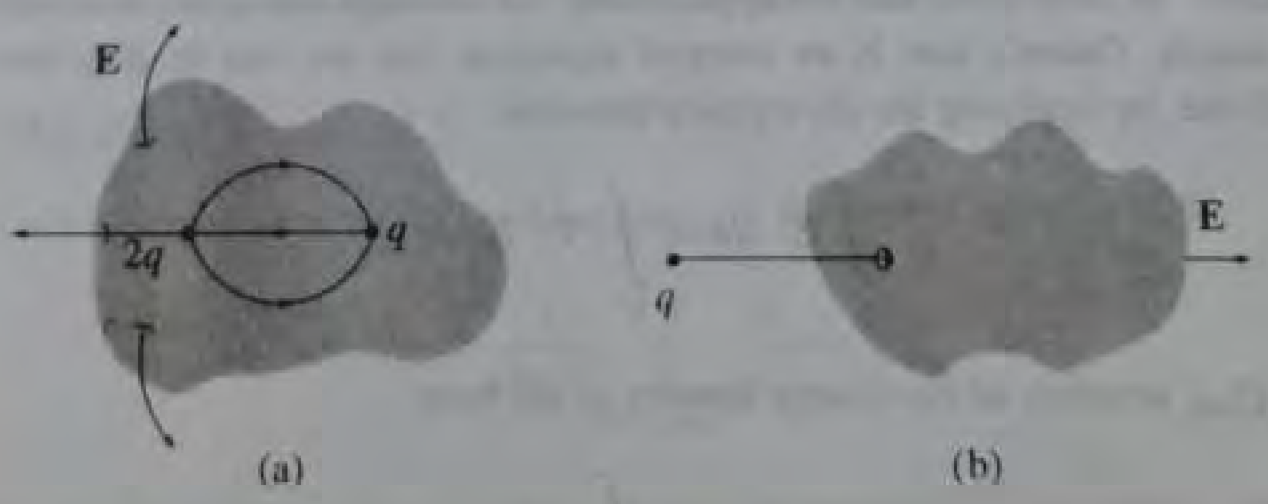


Figure 2.16

In the case of a point charge q at the origin, the flux of $\mathbf{E}$ through a sphere of radius r is

$$\oint \mathbf{E} \cdot d\mathbf{a} = \int \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r^2}\hat{\mathbf{r}}\right) \cdot (r^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}}) = \frac{1}{\epsilon_0}q. \tag{2.12}$$

Notice that the radius of the sphere cancels out, for while the surface area goes *up* as r^2, the field goes *down* as $1/r^2$, and so the product is constant. In terms of the field-line picture, this makes good sense, since the same number of field lines passes through any sphere centered at the origin, regardless of its size. In fact, it didn't have to be a sphere—any old surface, whatever its shape, would trap the same number of field lines. Evidently *the flux through any surface enclosing the charge is* q/ϵ_0.

Now suppose that instead of a single charge at the origin, we have a bunch of charges scattered about. According to the principle of superposition, the total field is the (vector) sum of all the individual fields:

$$\mathbf{E} = \sum_{i=1}^{n} \mathbf{E}_i.$$

The flux through a surface that encloses them all, then, is

$$\oint \mathbf{E} \cdot d\mathbf{a} = \sum_{i=1}^{n}\left(\oint \mathbf{E}_i \cdot d\mathbf{a}\right) = \sum_{i=1}^{n}\left(\frac{1}{\epsilon_0}q_i\right).$$

For any closed surface, then,

$$\boxed{\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}},} \tag{2.13}$$

where Q_{enc} is the total charge enclosed within the surface. This is the quantitative statement of Gauss's law. Although it contains no information that was not already present in Coulomb's law and the principle of superposition, it is of almost magical power, as you will see in Sect. 2.2.3. Notice that it all hinges on the $1/r^2$ character of Coulomb's law; without that the crucial cancellation of the r's in Eq. 2.12 would not take place, and the total flux of $\mathbf{E}$ would depend on the surface chosen, not merely on the total charge enclosed. Other $1/r^2$ forces (I am thinking particularly of Newton's law of universal gravitation) will obey "Gauss's laws" of their own, and the applications we develop here carry over directly.

As it stands, Gauss's law is an *integral* equation, but we can readily turn it into a *differential* one, by applying the divergence theorem:

$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\nabla \cdot \mathbf{E}) \, d\tau.$$

Rewriting Q_{enc} in terms of the charge density ρ, we have

$$Q_{\text{enc}} = \int_{\mathcal{V}} \rho \, d\tau.$$

So Gauss's law becomes

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{E})\, d\tau = \int_{\mathcal{V}} \left(\frac{\rho}{\epsilon_0}\right) d\tau.$$

And since this holds for *any* volume, the integrands must be equal:

$$\boxed{\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho.} \tag{2.14}$$

Equation 2.14 carries the same message as Eq. 2.13; it is **Gauss's law in differential form.** The differential version is tidier, but the integral form has the advantage in that it accommodates point, line, and surface charges more naturally.

Problem 2.9 Suppose the electric field in some region is found to be $\mathbf{E} = kr^3\hat{\mathbf{r}}$, in spherical coordinates (k is some constant).

(a) Find the charge density ρ.

(b) Find the total charge contained in a sphere of radius R, centered at the origin. (Do it two different ways.)

Problem 2.10 A charge q sits at the back corner of a cube, as shown in Fig. 2.17. What is the flux of $\mathbf{E}$ through the shaded side?

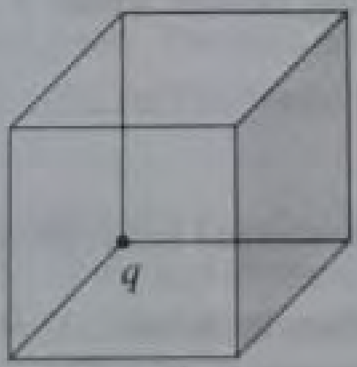


Figure 2.17

2.2.2 The Divergence of E

Let's go back, now, and calculate the divergence of $\mathbf{E}$ directly from Eq. 2.8:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\text{all space}} \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \rho(\mathbf{r}')\, d\tau'. \tag{2.15}$$

(Originally the integration was over the volume occupied by the charge, but I may as well extend it to all space, since $\rho = 0$ in the exterior region anyway.) Noting that the r-dependence is contained in $\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}'$, we have

$$\nabla \cdot \mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) \rho(\mathbf{r}')\, d\tau'.$$

This is precisely the divergence we calculated in Eq. 1.100:

$$\nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}}).$$

Thus

$$\nabla \cdot \mathbf{E} = \frac{1}{4\pi\epsilon_0} \int 4\pi\delta^3(\mathbf{r} - \mathbf{r}')\rho(\mathbf{r}')\, d\tau' = \frac{1}{\epsilon_0}\rho(\mathbf{r}), \tag{2.16}$$

which is Gauss's law in differential form (2.14). To recover the integral form (2.13), we run the previous argument in reverse—integrate over a volume and apply the divergence theorem:

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{E}\, d\tau = \oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} \int_{\mathcal{V}} \rho\, d\tau = \frac{1}{\epsilon_0} Q_{\text{enc}}.$$

2.2.3 Applications of Gauss's Law

I must interrupt the theoretical development at this point to show you the extraordinary power of Gauss's law, in integral form. When symmetry permits, it affords *by far* the quickest and easiest way of computing electric fields. I'll illustrate the method with a series of examples.

Example 2.2

Find the field outside a uniformly charged solid sphere of radius R and total charge q.

Solution: Draw a spherical surface at radius $r > R$ (Fig. 2.18); this is called a "Gaussian surface" in the trade. Gauss's law says that for this surface (as for any other)

$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}},$$

and $Q_{\text{enc}} = q$. At first glance this doesn't seem to get us very far, because the quantity we want (**E**) is buried inside the surface integral. Luckily, symmetry allows us to extract **E** from under the integral sign: **E** certainly points radially outward,[3] as does $d\mathbf{a}$, so we can drop the dot product,

$$\int_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \int_{\mathcal{S}} |\mathbf{E}|\, da,$$

[3] If you doubt that **E** is radial, consider the alternative. Suppose, say, that it points due east, at the "equator." But the orientation of the equator is perfectly arbitrary—nothing is spinning here, so there is no natural "north-south" axis—any argument purporting to show that **E** points east could just as well be used to show it points west, or north, or any other direction. The only *unique* direction on a sphere is *radial*.

Figure 2.18

and the *magnitude* of **E** is constant over the Gaussian surface, so it comes outside the integral:

$$\int_{\mathcal{S}} |\mathbf{E}|\, da = |\mathbf{E}| \int_{\mathcal{S}} da = |\mathbf{E}|\, 4\pi r^2.$$

Thus

$$|\mathbf{E}|\, 4\pi r^2 = \frac{1}{\epsilon_0} q,$$

or

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}.$$

Notice a remarkable feature of this result: The field outside the sphere is exactly the *same as it would have been if all the charge had been concentrated at the center.*

Gauss's law is always *true*, but it is not always *useful.* If ρ had not been uniform (or, at any rate, not spherically symmetrical), or if I had chosen some other shape for my Gaussian surface, it would still have been true that the flux of **E** is $(1/\epsilon_0)q$, but I would not have been certain that **E** was in the same direction as $d\mathbf{a}$ and constant in magnitude over the surface, and without that I could not pull $|\mathbf{E}|$ out of the integral. *Symmetry is crucial* to this application of Gauss's law. As far as I know, there are only three kinds of symmetry that work:

1. *Spherical symmetry.* Make your Gaussian surface a concentric sphere.
2. *Cylindrical symmetry.* Make your Gaussian surface a coaxial cylinder (Fig. 2.19).
3. *Plane symmetry.* Use a Gaussian "pillbox," which straddles the surface (Fig. 2.20).

Although (2) and (3) technically require infinitely long cylinders, and planes extending to infinity in all directions, we shall often use them to get approximate answers for "long" cylinders or "large" plane surfaces, at points far from the edges.

Figure 2.19

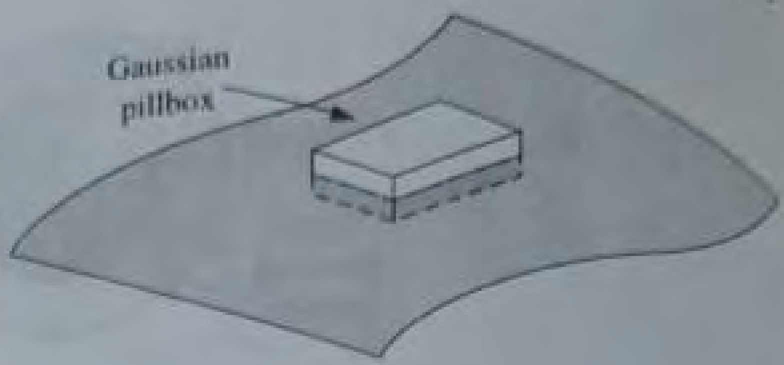


Figure 2.20

Example 2.3

A long cylinder (Fig. 2.21) carries a charge density that is proportional to the distance from the axis: $\rho = ks$, for some constant k. Find the electric field inside this cylinder.

Solution: Draw a Gaussian cylinder of length l and radius s. For this surface, Gauss's law states:

$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}.$$

The enclosed charge is

$$Q_{\text{enc}} = \int \rho\, d\tau = \int (ks')(s'\, ds'\, d\phi\, dz) = 2\pi k l \int_0^s s'^2\, ds' = \frac{2}{3}\pi k l s^3.$$

(I used the volume element appropriate to cylindrical coordinates, Eq. 1.78, and integrated ϕ from 0 to 2π, dz from 0 to l. I put a prime on the integration variable s', to distinguish it from the radius s of the Gaussian surface.)

Figure 2.21