March 20, 2020

Linearity of Conditional Expectation

Dr. SUSHIL KUMAR AZAD

Theorem 1. For any set A,
$$E(X+Y/A) = E(X/A) + E(Y/A).$$

Proof.
$$E(X+Y/A) = \sum_x \sum_y (x+y)\, p(X=x \,\&\, Y=y/A)$$
$$= \sum_x x \sum_y P(X=x \,\&\, Y=y/A)$$
$$+ \sum_y y \sum_x P(Y=y \,\&\, X=x/A)$$
$$= \sum_x x\, P(X=x/A)$$
$$+ \sum_y y\, P(Y=y/A).$$
$$\Rightarrow E(X+Y/A) = E(X/A) + E(Y/A). \quad //$$

SUSHIL KUMAR AZAD

Definition (Conditional Expectation). – The conditional expectation of Y given X, denoted by $E(Y|X)$, is a random variable that depends on X. It's value, when $X = x$, is $E(Y/X=x)$.

Theorem 2. Let (X, Y) be a random vector such that the variance of Y is finite. Then,

(a) $E[E(Y|X)] = E(Y)$

(b) $var[E(Y|X)] \leq var(Y)$.

Proof. We shall assume that X & Y are of continuous type random variables.

(a) Consider $E(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f(x, y)\, dy\, dx$

by definition

Continued –

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} y \, \frac{f(x,y)}{f_1(x)} dy \right] f_1(x)\, dx$$

$$= \int_{-\infty}^{\infty} E(Y/x) f_1(x)\, dx$$

$= E\left(E(Y/X) \right)$, which is the first result.

(b) Consider

$$Var(Y) = E[(Y-\mu_2)^2] \text{, by definition, where } \mu_2 = E(Y)$$

$$= E\{[Y - E(Y/X) + E(Y/X) - \mu_2]^2\}$$

$$= E\{[Y - E(Y/X)]^2\} + E\{[E(Y/X) - \mu_2]^2\}$$

$$+ 2E\{[Y - E(Y/X)][E(Y/X) - \mu_2]\} \quad \text{---- (i)}$$

Consider $E\{[Y - E(Y/X)][E(Y/X) - \mu_2]\}$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [y - E(Y/x)][E(Y/x) - \mu_2] f(x,y)\, dy\, dx$$

$$= \int_{-\infty}^{\infty} [E(Y/x) - \mu_2] \left\{ \int_{-\infty}^{\infty} [y - E(Y/x)] \frac{f(x,y)}{f_1(x)} dy \right\} f_1(x)\, dx$$

$$= \int_{-\infty}^{\infty} [E(Y/x) - \mu_2] \; E[(Y - E(Y/X))/X] f_1(x)\, dx$$

$$= \int_{-\infty}^{\infty} [E(Y/x) - \mu_2] \; [E(Y/X) - E(Y/X)] f_1(x)\, dx,$$

by Theorem 1 of Part (a)

$$= 0$$

$\therefore \; Var(Y) = E\{[Y - E(Y/X)]^2\} + E\{[E(Y/X) - \mu_2]^2\}$

$\geq E\{[E(Y/X) - \mu_2]^2\}$ as the first term is non-negative.

SUSHIL KUMAR AZAD

$$\Rightarrow \operatorname{var}(Y) \geq \operatorname{var}(E(Y|X)), \text{ as } E(E(Y|X)) = E(Y) = \mu_2,$$

which is the result. //

Example 17. Suppose a fair die is flipped. Let X denotes the number occurring on the top. Then a fair coin is tossed X times. If Y be the random variable which denotes the number of heads, find

(a) $E(Y/X)$

(b) $E(Y)$

Solution: The conditional probability for the $Y=y$, given $X=x$ is

$$P(Y=y/X=x) = \begin{cases} \binom{x}{y}\left(\frac{1}{2}\right)^x & \text{if } x \geq y \\ 0 & \text{else} \end{cases}$$

It can be seen that

$$P(Y=y/X=x) \sim \text{binomial}(x, \tfrac{1}{2})$$

Using the product rule, we can get the joint distribution of Y and X:

$$P(Y=y, X=x) = P(Y=y/X=x)\, P(X=x)$$

$$= \begin{cases} \frac{1}{6}\binom{x}{y}\left(\frac{1}{2}\right)^x & \text{if } x \geq y \\ 0 & \text{else.} \end{cases}$$

(a) Since $P(Y/X=x) \sim Bin(x, \frac{1}{2})$,

So $E(Y/X=x) = \frac{x}{2}$. (Mean of Binomial distribution $= np$)

Therefore, $\boxed{E(Y|X) = \frac{X}{2}}$

Continued ..

SUSHIL KUMAR AZAD

(b) Consider

$$E(Y) = E(E(Y|X)) \text{, by part (a) of theorem 2.}$$

$$= E\left(\frac{X}{2}\right) \text{, by first part}$$

$$= \frac{1}{2} E(X)$$

where $E(X) = (1+2+3+4+5+6) \times \frac{1}{6}$

$$= \frac{6 \cdot 7}{2 \cdot 1} \times \frac{1}{6} = \frac{7}{2} = 3.5$$

$$\therefore E(Y) = \frac{3.5}{2} = 1.75 \quad //$$

Note that we often abbreviate $E(Y/x)$ by $\mu_{Y/x}$.

SUSHIL KUMAR ATTRI

Example 18. Given the joint density

$$f(x,y) = \begin{cases} x e^{-x(1+y)} & x>0, y>0 \\ 0 & \text{elsewhere,} \end{cases}$$

find $\mu_{Y/x}$.

Solution. We first obtain the marginal pdf of X, $f_1(x)$ given by

$$f_1(x) = \int_0^\infty f(x,y)\,dy$$

$$= \int_0^\infty x e^{-x(1+y)}\,dy$$

$$= x e^{-x} \int_0^\infty e^{-xy}\,dy$$

$$= x \cdot e^{-x} \cdot \frac{e^{-xy}}{-x}\Bigg|_0^\infty = e^{-x}, \quad x>0$$

Continued ...

(5)

There, $f_1(x) = \begin{cases} e^{-x}, & x > 0 \\ 0 & \text{elsewhere} \end{cases}$

$\therefore$ The conditional density of Y, given $X = x$,

$f_{2|1}(Y|X=x)$, is

$$f_{2|1}(Y|x) = \frac{f(x,y)}{f_1(x)} = \begin{cases} x \cdot e^{-xy}, & y > 0 \\ 0 & \text{elsewhere}. \end{cases}$$

It can be seen that this conditional distribution, Y/X is an Exponential distribution with parameter $\theta = \frac{1}{x}$,

Now, if $Z \sim \exp(\theta)$ then $f(z,\theta) = \begin{cases} \frac{1}{\theta} e^{-z/\theta}, & z > 0 \\ 0 & \text{elsewhere} \end{cases}$

then we know, $\mu = E(Z) = \theta$

Therefore, $E(Y/x) = \frac{1}{x}$.

In fact, $E(Y/X=x) = \int_0^\infty y \cdot f_{2|1}(Y/x)\, dy$

$$= \int_0^\infty y \cdot x\, e^{-xy}\, dy$$

$$= \frac{1}{x}.$$

Hence, $\mu_{Y/x} \left(\equiv E(Y/X=x)\right)$

$$= \frac{1}{x}. \qquad //$$

SUSHIL KUMAR AZAD

Some more properties of conditional expectation:

Theorem 3. Let X, Y be random variables, and $g: \mathbb{R} \to \mathbb{R}$. Assuming all the following expectations exist, we have

(a) $E[X/Y] = E[X]$ if X and Y are independent.

(b) $E[X\, g(Y)/Y] = g(Y)\, E[X/Y]$.

In particular, $E[g(Y)|Y] = g(Y)$

(c) $E(X+Y/Y) = E(X|Y) + Y$

SUSHIL KUMAR AZAD

Proof. (a) We shall prove the continuous case. If X and Y are independent, then

$$f_{1|2}(x|y) = \frac{f(x,y)}{f_2(y)} = \frac{f_1(x)\, f_2(y)}{f_2(y)}$$

$$= f_1(x)$$

So, $$E[X|Y=y] = \int_{x \in S_1} x\, f_{1|2}(x|y)\, dx$$

$$= \int_{x \in S_1} x\, f_1(x)\, dx$$

$$\Rightarrow E[X|Y=y] = E[X].$$

(b) Given that $Y=y$, the possible values of $X g(Y)$ are $x\, g(y)$ where x varies over the range of X_y and the probability of the value $x\, g(y)$ given $Y=y$ is equal to $P(X=x|Y=y)$.

Continued

So, $E[X\,g(Y)|Y=y]$

$$= \sum_x x\,g(y)\,P(X=x|Y=y)$$

$$= g(y) \sum_x x\,P(X=x|Y=y)$$

$$= g(y)\,E[X|Y=y],$$ which is the result.

SUSHIL KUMAR AZAD

(c) $E(X+Y|Y)$

$$= E(X/Y) + E(Y/Y), \text{ by Theorem 1}$$

$$= E(X|Y) + Y, \text{ (by part (b), here } g(Y)=Y\text{.)}$$

Hence the result.

Note that the conditional expectation of X, given that $Y=y$, $E(X|Y=y)$, depends on the value of y. In other words, by changing y, $E[X|Y=y]$ can also change, that is, $E[X|Y=y]$ is a function of y.

So, let's write $g(y) = E[X|Y=y]$.

We can think this function $g(y)$, as a function of the value of of random variable Y. So, we can write

$$g(Y) = E[X|Y].$$

Thus, $E[X|Y]$ is a random variable whose value equals $g(y) = E[X|Y=y]$ when $Y=y$.

Example 19. Let $X = aY + b$. Then $E[X|Y] = aY + b$.

Solution. We have

$$E[X|Y=y] = E[aY+b \,|\, Y=y]$$

$$= ay + b, \quad \text{here } g(y) = ay+b, \text{ using part(b) of Theorem 3}$$

$$\Rightarrow E[X|Y] = aY + b.$$

Example 20. Let X and Y be two random variables, and g and h be two functions. Then

$$E[g(X)h(Y)|X] = g(X)\,E[h(Y)|X].$$

Solution. We have, (considering the continuous case)

$$E[g(X)h(Y)|X=x]$$

$$= \int_y g(x)h(y) f_{2|1}(y|x)\,dy$$

$$= g(x)\int_y h(y) f_{2|1}(y|x)\,dy$$

$$= g(x)\,E[h(Y)|X=x]$$

$$\Rightarrow E[g(X)h(Y)|X] = g(X)\,E[h(Y)|X].$$

//

SUSHIL KUMAR AZAD

Example 20. Let the joint pdf of X and Y be given by

$$f(x,y)=\begin{cases}\dfrac{2}{(1+x+y)^3}, & 0<x<\infty,\ 0<y<\infty\\ 0 & \text{elsewhere.}\end{cases}$$

(a) Compute the marginal pdf of X, and the conditional pdf of Y, given $X=x$.

(b) For a fixed $X=x$, compute $E(1+x+Y/x)$, and use the result to compute $E(Y|x)$.

SUSHIL KUMAR AZAD

Solution. (a) The marginal pdf of X, $f_1(x)$, is

$$f_1(x)=\int_0^\infty f(x,y)\,dy$$

$$=\int_0^\infty \frac{2}{(1+x+y)^3}\,dy$$

$$=2\left|\frac{(1+x+y)^{-2}}{-2}\right|_0^\infty=\frac{1}{(1+x)^2},\quad 0<x<\infty$$

$$\therefore\quad f_1(x)=\begin{cases}\dfrac{1}{(1+x)^2}, & 0<x<\infty\\ 0 & \text{elsewhere}\end{cases}\qquad \text{---(i)}$$

So, the conditional pdf of Y, given $X=x$, is

$$f_{2|1}(y/x)=\frac{f(x,y)}{f_1(x)},\quad x\in S_1$$

$$=\frac{2(1+x)^2}{(1+x+y)^3}\quad \text{if } 0<x<\infty,\ 0<y<\infty$$

(b) $$E(1+x+Y/x)=\int_0^\infty (1+x+y)\,f_{2|1}(y|x)\,dy$$

$$=\int_0^\infty \frac{2(1+x)^2}{(1+x+y)^2}\,dy=2(1+x)^2\times\frac{1}{(1+x)}$$

$$\Rightarrow E(1+x+Y/x)=2(1+x),\quad 0<x<\infty\qquad \text{---(ii)}$$

(10)

So, $E(1+X+Y/X) = 2(1+X)$

Now, $E(1+X+Y/X)$

$$= E[(1+Y)/X] + E[X/X]$$

$$= E[1/X] + E[Y/X] + X$$

where. $E(1/X=x) = \int_0^\infty 1 \, f_{2/1}(y/x)\, dy$

$$= 1$$

$$\Rightarrow E(1+X+Y/X) = 1 + E(Y/X) + X$$

$$\Rightarrow 2(1+X) = 1 + E(Y/X) + X$$

$$\Rightarrow E(Y/X) = 1$$

$\Rightarrow E[Y|X=x] = 1$, that the function, $g(x) = E[Y|X=x]$ is a constant function.

Sahil Kumar Azad