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Independent Random Variables

Definition 1: Let X_1 and X_2 be the random variables having the joint pdf $f(x_1, x_2)$ (joint pmf $p(x_1, x_2)$). Then the random variables X_1 and X_2 are said to be *independent* if, and only if,

$$f(x_1, x_2) = f_{X_1}(x_1)\, f_{X_2}(x_2)$$

$$\left(p(x_1, x_2) = p_{X_1}(x_1)\, p_{X_2}(x_2) \right),$$

where $f_{X_1}(x_1)$ $(p_{X_1}(x_1))$ and $f_{X_2}(x_2)$ $(p_{X_2}(x_2))$ are the marginal pdfs (pmfs) of X_1 & X_2 respectively.

Theorem 1 When the conditional pdf $f_{2|1}(x_2|x_1)$ does not depend upon x_1, we have

$$f_2(x_2) = f_{2|1}(x_2|x_1), \text{ and}$$

$$f(x_1, x_2) = f_1(x_1)\, f_2(x_2).$$

Solution: We know that the marginal pdf of X_2 is, for random variables of the continuous type,

$$f_2(x_2) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_1$$

Since, $f_{2|1}(x_2|x_1) = \dfrac{f(x_1, x_2)}{f_1(x_1)}$, we have

$$f_2(x_2) = \int_{-\infty}^{\infty} f_{2|1}(x_2|x_1)\, f_1(x_1)\, dx_1$$

$$= f_{2|1}(x_2|x_1) \int_{-\infty}^{\infty} f_1(x_1)\, dx_1$$

$$\Rightarrow f_2(x_2) = f_{2|1}(x_2|x_1) \quad \text{as} \quad \int_{-\infty}^{\infty} f_1(x_1)\, dx_1 = 1$$

Continued

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Also, $f(x_1, x_2) = f_{2/1}(x_2/x_1)\, f_1(x_1)$

$$= f_2(x_1)\, f_1(x_1)$$

Thus, if the conditional distribution of X_2, given $X_1 = x_1$, is independent of any assumption about x_1, then $f(x_1, x_2) = f(x_1) f(x_2)$, that i.e., the random variables X_1 and X_2 are independent. //

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Example 8. If the random variables X_1 and X_2 have the joint pdf $f(x_1, x_2) = \begin{cases} 2e^{-x_2 - x_1}, & 0 < x_1 < x_2, 0 < x_2 < \infty \\ 0, & \text{elsewhere,} \end{cases}$

then X_1 and X_2 are dependent.

Solution: The marginal pdfs of X_1 & X_2 are

$$f_1(x_1) = \begin{cases} \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_2 = 2\int_{x_1}^{\infty} e^{-x_2} e^{-x_1}\, dx_2 \\ = 2e^{-x_1} e^{-x_1} = 2e^{-2x_1}, & 0 < x_1 < x_2,\ 0 < x_2 < \infty \\ 0 & \text{elsewhere} \end{cases}$$

and $$f_2(x_2) = \begin{cases} \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_1 = 2e^{-x_2} \int_0^{x_2} e^{-x_1}\, dx_1 \\ = 2e^{-2x_2}, & 0 < x_2 < \infty \\ 0 & \text{elsewhere} \end{cases}$$

We have, $$f_1(x_1) f_2(x_2) = \begin{cases} 2e^{-2(x_1 + x_2)}, & 0 < x_1 < x_2,\ x_2 < \infty \\ 0 & \text{elsewhere} \end{cases}$$

$$\neq f(x_1, x_2),$$

$\therefore$ the random variables X_1 and X_2 are dependent. //

Theorem 2. Let the random variables X_1 and X_2 have supports S_1 and S_2, respectively, and have the joint pdf $f(x_1, x_2)$. Then X_1 and X_2 are independent if, and only if

$$f(x_1, x_2) = g(x_1)\, h(x_2), \quad \text{---- (i)}$$

for some nonnegative function, $g(x_1)$, of x_1 and a nonnegative function, $h(x_2)$, of x_2, that is, $g(x_1) > 0$ when $x_1 \in S_1$ and zero elsewhere, and $h(x_2) > 0$ when $x_2 \in S_2$ and zero elsewhere.

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Proof. Let X_1 and X_2 are independent random variables. Then, we have

$$f(x_1, x_2) = f_1(x_1)\, f_2(x_2).$$

where $f_1(x_1)$ and $f_2(x_2)$ are the marginal pdfs of X_1 & X_2, respectively.

$\therefore$ By the def^n of pdf,

$f_1(x_1) > 0$ when $x_1 \in S_1$ & zero elsewhere

and $f_2(x_2) > 0$ when $x_2 \in S_2$ & zero elsewhere

Thus, the condition (i) is fulfilled.

Conversely, let the condition (i) is fulfilled, that is, $f(x_1, x_2) = g(x_1)\, h(x_2)$

Now $f_1(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_2 = \int_{-\infty}^{\infty} g(x_1)\, h(x_2)\, dx_2$

$= g(x_1) \int_{-\infty}^{\infty} h(x_2)\, dx_2 = C_1\, g(x_1),$

where $C_1 = \int_{-\infty}^{\infty} h(x_2)\, dx_2$ is a constant.

Continued ...

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Next, $f_2(x_2) = \int_{-\infty}^{\infty} f(x_1, x_2)\,dx_1$

$= h(x_2) \int_{-\infty}^{\infty} g(x_1)\,dx_1$

$= c_2\, h(x_2)$, where $c_2 = \int_{-\infty}^{\infty} g(x_1)\,dx_1$ is a constant

By defn of pdf, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2)\,dx_1\,dx_2 = 1$$

$$\Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1)\,h(x_2)\,dx_1\,dx_2 = 1$$

$$\Rightarrow \left(\int_{-\infty}^{\infty} g(x_1)\,dx_1\right)\left(\int_{-\infty}^{\infty} h(x_2)\,dx_2\right) = 1$$

$$\Rightarrow \quad c_1\, c_2 = 1$$

Thus, $f(x_1, x_2) = g(x_1)\,h(x_2)$

$= c_1 c_2\, g(x_1)\,h(x_2)$

$= c_1\, g(x_1) \cdot c_2\, h(x_2)$

$\Rightarrow f(x_1, x_2) = f_1(x_1)\, f_2(x_2)$

$\Rightarrow$ X_1 and X_2 are independent random variables. //

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Example 8. Find $P(0<X_1<\frac{1}{3}, 0<X_2<\frac{1}{3})$ if the random variables X_1 and X_2 have the joint pdf $f(x_1, x_2) = 4x_1(1-x_2)$, $0<x_1<1$, $0<x_2<1$, zero elsewhere.

Solution. We have

$$f(x_1, x_2) = \begin{cases} 4x_1(1-x_2), & 0<x_1<1, \; 0<x_2<1 \\ 0 & \text{elsewhere} \end{cases}$$

Let

$$g(x_1) = \begin{cases} 2x_1 & 0<x_1<1 \\ 0 & \text{elsewhere} \end{cases}, \text{ and}$$

$$h(x_2) = \begin{cases} 2(1-x_2), & 0<x_2<1 \\ 0 & \text{elsewhere} \end{cases}$$

Then $f(x_1, x_2) = g(x_1)\, h(x_2)$, where

$g(x_1) > 0$ if $x_1 \in S_1$ & $h(x_2) > 0$ if $x_2 \in S_2$

Hence, X_1 and X_2 are independent random variables.

$$\Rightarrow P(0<X_1<\tfrac{1}{3}, 0<X_2<\tfrac{1}{3})$$

$$= \left(\int_0^{1/3} 2x_1\, dx_1\right)\left(\int_0^{1/3} 2(1-x_2)\, dx_2\right) \quad \left(\begin{array}{c}\text{See} \\ \text{Theorem 4} \\ \text{on page (7)}\end{array}\right)$$

$$= \left(\tfrac{1}{3}\right)^2 \left[1-\left(\tfrac{2}{3}\right)^2\right] = \tfrac{5}{81} \qquad //$$

Theorem 3. Let $F(x_1, x_2)$ be the joint cdf of random vector (X_1, X_2), and $F_1(x_1)$ $(F_2(x_2))$ be the marginal cdf of X_1 (X_2), respectively. Then X_1 and X_2 are independent if and only if

$$F(x_1, x_2) = F_1(x_1)\, F_2(x_2) \quad \forall (x_1, x_2) \in \mathbb{R}^2$$

Continued.

Proof. Let X_1 and X_2 be independent random variables.

Then $f(x_1, x_2) = f_1(x_1) f_2(x_2)$ ------ (i)

By definition, $F(x_1, x_2) = P(X_1 \le x_1, X_2 \le x_2)$

$$= \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} f(w_1, w_2)\, dw_2\, dw_1$$

$$= \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} f_1(w_1)\, f_2(w_2)\, dw_2\, dw_1 \quad \text{by (i)}$$

$$= \left(\int_{-\infty}^{x_1} f_1(w_1)\, dw_1\right)\left(\int_{-\infty}^{x_2} f_2(w_2)\, dw_2\right)$$

$$= F_1(x_1)\, F_2(x_2).$$

$$\therefore\ F(x_1, x_2) = F_1(x_1)\, F_2(x_2).$$

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Conversely, Suppose

$$F(x_1, x_2) = F_1(x_1)\, F_2(x_2).$$

Then,

$$f(x_1, x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F(x_1, x_2)$$

$$= \frac{\partial^2}{\partial x_1 \partial x_2} F_1(x_1) F_2(x_2)$$

$$= \frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_2} F_1(x_1) F_2(x_2)\right)$$

$$= \frac{\partial}{\partial x_1}\left(F_1(x_1) \frac{\partial}{\partial x_2} F_2(x_2)\right)$$

$$= \frac{\partial}{\partial x_1}\left(F_1(x_1) f_2(x_2)\right)$$

$$= f_2(x_2) \frac{\partial}{\partial x_1} F_1(x_1) = f_2(x_2) f_1(x_1)$$

$\therefore\ f(x_1, x_2) = f_1(x_1) f_2(x_2)$; hence X_1 and X_2 are independent. //

Theorem 4. The random variables X_1 and X_2 are independent random variables if, and only if the following condition holds:

$$P(a < X_1 \leq b,\ c < X_2 \leq d) = P(a < X_1 \leq b)\, P(c < X_2 \leq d) \quad \text{---(ii)}$$

Proof: Let X_1 and X_2 be independent. Then

$$P(a < X_1 \leq b,\ c < X_2 \leq d)$$
$$= F(b, d) - F(a, d) - F(b, c) + F(a, c)$$
$$= F_1(b)F_2(d) - F_1(a)F_2(d) - F_1(b)F_2(c) + F_1(a)F_2(c)$$
$$= [F_1(b) - F_1(a)][F_2(d) - F_2(c)] \quad \text{(using Theorem 3)}$$
$$= P(a < X_1 \leq b)\, P(c < X_2 \leq d)$$

Conversely, suppose the condition (ii) holds.

Consider
$$F(x_1, x_2) = P(-\infty < X_1 \leq x_1,\ -\infty < X_2 \leq x_2)$$
$$= P(-\infty < X_1 \leq x_1)\, P(-\infty < X_2 \leq x_2) \quad \text{using (ii)}$$

$$\Rightarrow F(x_1, x_2) = F_1(x_1)\, F_2(x_2)$$

Hence, using Theorem 3, it implies that X_1 and X_2 are independent.

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Example 10. Let the joint pdf of X_1 and X_2 be

$$f(x_1, x_2) = \begin{cases} x_1 + x_2, & 0 < x_1 < 1,\ 0 < x_2 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Show that X_1 and X_2 are dependent.

Solution: We have

$$P\left(0 < X_1 \leq \tfrac{1}{2},\ 0 < X_2 \leq \tfrac{1}{2}\right) = \int_0^{1/2}\int_0^{1/2} (x_1 + x_2)\, dx_1 dx_2 = \frac{1}{8},$$

continued ...

and the marginal pdfs of X_1 and X_2 are

$$f_1(x_1) = \begin{cases} \int_0^1 (x_1+x_2)\,dx_2 = x_1 + \frac{1}{2}, & 0 < x_1 < 1 \\ 0 & \text{elsewhere}, \end{cases}$$

$$f_2(x_2) = \begin{cases} \int_0^1 (x_1+x_2)\,dx_1 = \frac{1}{2} + x_2, & 0 < x_2 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Now, $P(0 < X_1 \le \frac{1}{2}) = \int_0^{1/2} f_1(x_1)\,dx_1$

$$= \int_0^{1/2} (x_1 + \tfrac{1}{2})\,dx_1 = \frac{3}{8},$$

and $P(0 < X_2 \le \frac{1}{2}) = \int_0^{1/2} (\frac{1}{2} + x_2)\,dx_2$

$$= \frac{3}{8}.$$

Since, the condition (ii) does not hold, X_1 and X_2 are dependent. //

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Theorem 5. Suppose X_1 and X_2 are independent, and that $E(u(X_1))$ and $E(v(X_2))$ exist. Then

$$E[u(X_1)v(X_2)] = E[u(X_1)]\,E[v(X_2)]$$

Proof: Since X_1 and X_2 are independent,

$$f(x_1, x_2) = f_1(x_1) f_2(x_2).$$

By definition,

$$E[u(X_1)v(X_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x_1)\,v(x_2)\,f_1(x_1)f_2(x_2)\,dx_1\,dx_2$$

$$= \left[\int_{-\infty}^{\infty} u(x_1)f_1(x_1)\,dx_1\right]\left[\int_{-\infty}^{\infty} v(x_2)f_2(x_2)\,dx_2\right]$$

$$= E[u(X_1)]\,E[v(X_2)].$$ //

We now prove a very useful theorem about independent random variables.

<u>Theorem 6.</u> Suppose the joint mgf, $M(t_1, t_2)$ exists for the random variables X_1 and X_2. Then X_1 and X_2 are <u>independent</u> if, and only if

$$M(t_1, t_2) = M(t_1, 0)\, M(0, t_2) \quad \text{---(iii)}$$

<u>Proof</u>: Let X_1 and X_2 be independent.

By definition,

$$M(t_1, t_2) = E\left[e^{t_1X_1 + t_2X_2} \right]$$
$$= E\left[e^{t_1X_1} e^{t_2X_2} \right]$$
$$= E\left[e^{t_1X_1} \right] E\left[e^{t_2X_2} \right], \text{ using Theorem 5.}$$

$$\Rightarrow M(t_1, t_2) = M(t_1, 0)\, M(0, t_2)$$

$\Rightarrow$ the condition (iii) holds.

Conversely, Suppose the condition (iii) holds.

By definition.

$$M(t_1, 0) = E\left(e^{t_1X_1} \right)$$
$$= \int_{-\infty}^{\infty} e^{t_1x_1} f_1(x_1)\, dx_1, \text{ and}$$

$$M(0, t_2) = E\left(e^{t_2X_2} \right)$$
$$= \int_{-\infty}^{\infty} e^{t_2x_2} f_2(x_2)\, dx_2$$

Thus, we have

$$M(t_1, 0)\, M(0, t_2) = \left[\int_{-\infty}^{\infty} e^{t_1x_1} f_1(x_1)\, dx_1 \right] \left[\int_{-\infty}^{\infty} e^{t_2x_2} f_2(x_2)\, dx_2 \right]$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{t_1x_1 + t_2x_2} f_1(x_1) f_2(x_2)\, dx_1\, dx_2$$

continued ...

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Now, $M(t_1, t_2) = M(t_1, 0)\, M(0, t_2)$ by assumption

$$\Rightarrow M(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1x_1+t_2x_2} f_1(x_1) f_2(x_2)\, dx_1\, dx_2 \quad \text{--- (iv)}$$

but, by def^n. $M(t_1, t_2) = E\left[e^{t_1X_1+t_2X_2}\right]$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1x_1+t_2x_2} f(x_1, x_2)\, dx_1\, dx_2 \quad \text{--- (v)}$$

Comparing (iv) & (v), we get

$$f(x_1, x_2) = f_1(x_1) f_2(x_2),$$

Since the mgf of X_1 & X_2, $M(t_1, t_2)$ is unique.

$\Rightarrow X_1$ and X_2 are independent. //

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Example 11. Let X_1 and X_2 be random variables with joint pmf $p(x_1, x_2) = \dfrac{1}{2^{x_1+x_2}}$, for $1 \le x_i < \infty$, $i = 1, 2$, where x_1 and x_2 are integers, zero elsewhere. Determine the joint mgf of (X_1, X_2) and, show that X_1 and X_2 are independent random variables.

Solution. By definition the mgf of X_1 and X_2,

$$M(t_1, t_2) = E\left[e^{t_1X_1+t_2X_2}\right]$$

$$= \sum_{x_1=1}^{\infty}\sum_{x_2=1}^{\infty} e^{t_1x_1+t_2x_2} \cdot \frac{1}{2^{x_1+x_2}}$$

Continued . .

$$= \sum_{x_1=1}^{\infty} \sum_{x_2=1}^{\infty} \left[e^{t_1 x_1} \left(\tfrac{1}{2}\right)^{x_1} e^{t_2 x_2} \left(\tfrac{1}{2}\right)^{x_2} \right]$$

$$= \sum_{x_1=1}^{\infty} \left[\frac{e^{t_1}}{2}\right]^{x_1} \sum_{x_2=1}^{\infty} \left[\frac{e^{t_2}}{2}\right]^{x_2}$$

$$= \left[\frac{\left(\frac{e^{t_1}}{2}\right)}{1-\left(\frac{e^{t_1}}{2}\right)}\right] \left[\frac{\left(\frac{e^{t_2}}{2}\right)}{1-\left(\frac{e^{t_2}}{2}\right)}\right]$$

$$\Rightarrow M(t_1, t_2) = \left[\frac{e^{t_1}}{(2-e^{t_1})}\right] \left[\frac{e^{t_2}}{2-e^{t_2}}\right],$$

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Now, the marginal pmfs of X_1 and X_2 are,

$$p_1(x_1) = \sum_{x_2=1}^{\infty} p(x_1, x_2)$$

$$= \frac{1}{2^{x_1}} \sum_{x_2=1}^{\infty} \left(\frac{1}{2}\right)^{x_2}$$

$$= \frac{1}{2^{x_1}}, \quad 1 \le x_1 < \infty$$

$$\& \quad p_2(x_2) = \frac{1}{2^{x_2}}, \quad 1 \le x_2 < \infty.$$

Thus, $M(t_1, 0) = E\left[e^{tX_1}\right]$

$$= \sum_{x_1=1}^{\infty} e^{t_1 x_1} p_1(x_1)$$

$$= \sum_{x_1=1}^{\infty} e^{t_1 x_1} \cdot \frac{1}{2^{x_1}} = \sum_{x_1=1}^{\infty} \left(\frac{e^{t_1}}{2}\right)^{x_1}$$

$$= \frac{e^{t_1}}{(2-e^{t_1})},$$

continued...

Similarly, $M(0,t_2) = \dfrac{e^{t_2}}{(2-e^{t_2})}$.

Thus, $M(t_1,0)\,M(0,t_2)$

$$= \left(\frac{e^{t_1}}{2-e^{t_1}}\right)\left(\frac{e^{t_2}}{2-e^{t_1}}\right)$$

$$= M(t_1,t_2)$$

$\Rightarrow$ X_1 and X_2 are independent.

Example 12. Let X and Y have the joint pdf

$$f(x,y) = \begin{cases} 3x, & 0<y<x<1 \\ 0, & \text{elsewhere} \end{cases}$$

Are X and Y independent? If not, find $E(X|y)$.

Solution. The marginal pdfs of X and Y are

$$f_X(x) = \int_0^x f(x,y)\,dy$$

$$= \int_0^x 3x\,dy = 3x^2$$

$$\therefore f_X(x) = \begin{cases} 3x^2, & 0<x<1 \\ 0 & \text{elsewhere} \end{cases},$$

and $f_Y(y) = \int_y^1 3x\,dx = \begin{cases} \frac{3}{2}(1-y^2), & 0<y<x<1 \\ 0, & \text{elsewhere} \end{cases}$

Now, $f_X(x)\,f_Y(y) = \begin{cases} \frac{9}{2}x^2(1-y^2), & 0<y<x<1 \\ 0, & \text{elsewhere} \end{cases}$

which is not equal to $f(x,y)$, implies that X and Y are dependent.

continued...

Next $f_{1/2}(x/y) = \dfrac{f(x,y)}{f_y(y)}, \quad y \in S_Y$

$$= \frac{3x}{\left(\frac{3}{2}\right)(1-y^2)} = \frac{2x}{(1-y^2)}$$

$\therefore \quad E(X|y) = \displaystyle\int_y^1 x \, f_{1|2}(x|y)\, dx$

$$= \int_y^1 \frac{2x^2}{(1-y^2)}\, dx = \frac{2}{(1-y^2)} \times \frac{1}{3}(1-y^3)$$

$\Rightarrow \quad E(X|y) = \dfrac{2(1+y+y^2)}{3(1+y)}$ //

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