April 09, 2020
@Dr. SUSHIL KUMAR AZAD

A Random Walk Model

Definition 1. A Markov chain whose state space is given by the integers $i = 0, \pm 1, \pm 2, \ldots$ is said to a random walk if, for some number $0 < p < 1$,

$$P_{i,i+1} = p, \text{ and}$$
$$P_{i,i-1} = 1-p.$$

The transition probability matrix is

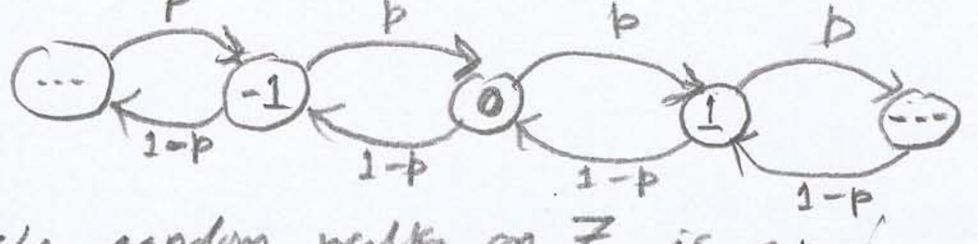
$$P = \begin{Vmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1-p & 0 & p & 0 & 0 & \cdots \\ \cdots & 0 & 1-p & 0 & p & 0 & \cdots \\ \cdots & 0 & 0 & 1-p & 0 & p & \cdots \\ \cdots & 0 & 0 & 0 & 1-p & 0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{Vmatrix}$$

We call it a random walk, because we may think of it as being a model for "an individual walking on a straight line who at each point of time either takes one step to right with probability, p or one step to the left with probability, 1-p".

Definition 2. The probability that, the Markov chain, starting in state i, eventually returns to state i, denoted by f_i, is given by

$$f_i = P\{X_n = i \text{ for some } n \geq 1 \mid X_0 = i\}$$

The transition graph of random walk is



The <u>random walk on $\mathbb{Z}$</u> is an example of a Markov chain on <u>a countably infinite state space</u>.

TUSHIL KUMAR AZAD

Clearly, for $0 < p < 1$, there is only one communication (or equivalence) class, a state space itself.

$\Rightarrow$ the <u>random walk is irreducible</u>.

We shall now classify the states into two classes, viz., <u>recurrent</u> and <u>transient</u>.

<u>Definition 3</u>. We say that state i is <u>recurrent</u> if $f_i = 1$.

If $f_i < 1$, state i is called <u>transient</u>.

<u>Theorem 1</u>. If state i is recurrent, then starting in state i, the process will reenter state i infinitely often.

<u>Proof</u>. Suppose that the process starts in state i and state i is recurrent.

continued..

Now $f_i = 1$, and the process will eventually reenter state i.

By the definition of MC, it follows that the process will be starting over again when it reenters state i.

$\Rightarrow$ state i will eventually be visited again

Therefore, we, repeating the above argument again & again, conclude that if state i is recurrent, then, starting in state i, the process will reenter state i again, infinitely often. //

SUSHIL KUMAR AZAD

Theorem 2. If state i is transient, then, starting in state i, the number of time periods that the process will be in state i, has a geometric distribution with finite mean $= \dfrac{1}{(1-f_i)}$.

Proof. Suppose $X_0 = i$, and N denote the number of times that MC is in state i (before leaving i forever).

Clearly, $N \geq 1$, since $X_0 = i$.

Now $P\{N = 1\} = 1 - f_i$ (never returns)

Continued —

$P\{N=2\} = f_i(1-f_i)$ (returns exactly once)

— — —

In general, we get

$$P\{N=k\} = f_i^{k-1}(1-f_i)$$

(returns $(k-1)$-times)

So, $N \sim Geom(p)$, where $p = 1-f_i$, that is, N is a geometric variate with parameter, p.

Since $f_i < 1$, we have

$$E[N] = \frac{1}{p} = \frac{1}{1-f_i} < \infty. \quad //$$

Corollary 1. State i is <u>recurrent</u> if, and only if, starting in state i, the expected number of time periods that the process is in state i is infinite, i.e., $\boxed{E[N] = \infty}$

<u>Proof.</u> If state i is recurrent, then by Theorem 1, the MC returns to state i an infinite number of times.

$\Rightarrow E[N] = \infty.$

— . . —

<u>Theorem 3.</u> State i is <u>recurrent</u> iff

$$\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty, \quad \text{and}$$

state i is transient iff

$$\sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty.$$

SUSHIL KUMAR AZAD

<u>Proof.</u> Define the event

$$I_n = \begin{cases} 1 & \text{if } X_n = i \\ 0 & \text{if } X_n = 0 \end{cases}$$

Then, the number of time periods that the process will return to state i, N, is

$$N = \sum_{n=1}^{\infty} I_n \qquad \text{------} (*)$$

Let state i be recurrent. Then, we have

$$\sum_{n=1}^{\infty} P_{ii}^{(n)} = \sum_{n=1}^{\infty} P\{X_n = i / X_0 = i\}$$

$$= \sum_{n=1}^{\infty} E[I_n / X_0 = i],$$

treating the expected values of an indication function, I_n, as a probability

$$= E\left[\sum_{n=1}^{\infty} I_n \,\middle|\, X_0 = i\right]$$

$$= E[N \mid X_0 = i], \quad (N = \text{number of returns} = \sum_{n=1} I_n, \text{ from } (*))$$

$$= E[N] \quad (X_0 = i)$$

$$= \infty, \text{ by Corollary}$$

In Case, state i is transient, then we Know that $N \sim \text{Geom}(1-f_i)$, and

$$E[N] = \frac{1}{1-f_i}, \quad f_i < 1$$
$$< \infty$$

Since, $N = \sum_{n=1}^{\infty} P_{ii}^{(n)}$.

So, State i is transient

iff $E[N] < \infty$

iff $\sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty$. //

SUSHIL KUMAR AZAD

Corollary 2. If state i is recurrent (transient), and $i \Leftrightarrow j$, then j is recurrent (transient)

Proof see Corollary 4.2, page 197, Ross

Corollary 3. In a MC with a finite number of states, not all of the states can be transient.

Proof. Suppose the states are $\{0, 1, 2, ---, M\}$, and that they are all transient.

Then, after a finite amount of time, say, after time T_0, state 0 will never be visited, since by Theorem 3, if state i is transient, then it will only be visited a finite number of times

Continued..

Similarly, state 1, being transient, after a time, say T_1, will never be visited, and state 2 after a time, say T_2, will never be visited and so on.

Let $T = \max\{T_0, T_1, \dots, T_M\}$.

Then, after a finite time, T, no states will be visited.

But, by the definition of MC, the process must be in some state after time, T, $\Rightarrow$ a contradiction to our assumption

$\Rightarrow$ at least one of the states must be recurrent. //

SUSHIL KUMAR AZAD

Corollary 4. All states of a finite irreducible MC are recurrent.

Since, if the MC is irreducible, there is only one communication class, and by Corollary 3, there is at least one recurrent state in MC, and by Corollary 2, if $i \leftrightarrow j$ then j is recurrent.

$\Rightarrow$ all states are recurrent. //

<u>Definition 4</u>. By Corollary 2, all states in an equivalence class are recurrent (transient) if one state in that class is recurrent (transient). Such a class is called a recurrent equivalence class (transient equivalence class).

SUSHIL KUMAR AZAD

<u>Example 46</u> Consider a 2-state MC with transition probability matrix & transition graph:

$$P = \begin{array}{c} 0 \\ 1 \end{array} \begin{Vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{Vmatrix}$$

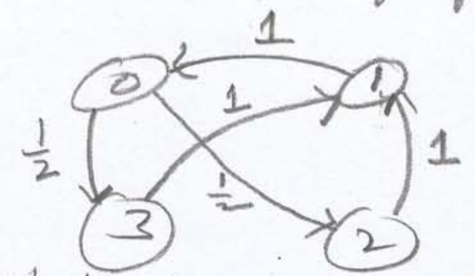
Clearly, all states communicate. So, this is a <u>finite irreducible MC</u>. Therefore, by Corollary 4, all states are recurrent.

<u>Example 47</u>. Consider the prob. transition matrix

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 \end{array}$$

Determine which states are transient and which are recurrent.

Solution, The transition graph of MC is



Consider state 0.

We have 2-possible cycles/paths when state 0 eventually reenters, viz.,

0 − 2 − 1 − 0 , and

0 − 3 − 1 − 0

So, $P_{00}^{(3)} = \frac{1}{2} \times 1 \times 1 + \frac{1}{2} \times 1 \times 1$

$= 1$

$\Rightarrow$ state 0 is recurrent.

Next, we have

(i) $1 \xrightarrow{1} 0$ and $0 \xrightarrow{\frac{1}{2}} 2 \xrightarrow{1} 1$

$\Rightarrow 0 \Leftrightarrow 1$

(ii) $2 \xrightarrow{1} 1 \xrightarrow{1} 0$ and $0 \xrightarrow{\frac{1}{2}} 2$

$\Rightarrow 0 \Leftrightarrow 2$, and

(iii) $0 \xrightarrow{\frac{1}{2}} 3$ and $3 \xrightarrow{1} 1 \xrightarrow{1} 0$

$\Rightarrow 0 \Leftrightarrow 3$

$\therefore$ by Corollary 2, states 1, 2 & 3 are also recurrent

$\Rightarrow$ All states are recurrent. //

SUSHIL KUMAR AZAD

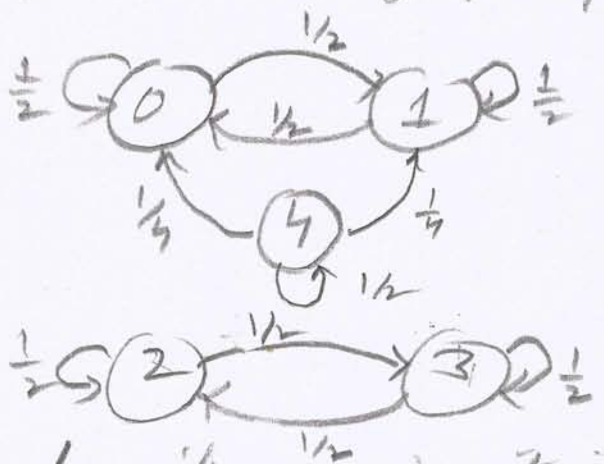
<u>Example 48.</u> Consider the MC having states 0, 1, 2, 3, 4 and

$$P = \begin{Vmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{Vmatrix}$$

Determine the recurrent state.

SUNIL KUMAR AZAD

<u>Solution.</u> The transition graph of given MC is



From the transition graph, it is clear that the given MC consists of 3 equivalence classes, viz.,

$$\{0, 1\},\ \{2, 3\},\ \{4\}.$$

It can also be seen that state 0 can reenter infinitely often, and so is the state 2.

$\Rightarrow$ state 0 and state 2 are recurrent.

$\Rightarrow$ $\{0, 1\}$ and $\{2, 3\}$ are recurrent equivalence classes.

continued...

But the state 4 is transient, since $f_4 = \frac{1}{2}$, and if,

N = number of times that state 4 eventually reenters, then

$$N \sim Geom\left(\tfrac{1}{2}\right),$$

$$\text{so, } E(N) = \frac{1}{(1/2)} = 2 < \infty.$$

//

SUSHIL KUMAR AZAD

Example 4.9. In a random walk, determine which states are transient and which are recurrent.

Solution. We have already shown (page 2) that all states of a random walk communicate.

So, by Corollary 2, all states are either transient or all are recurrent.

So let us consider state 0 & determine if $\sum_{n=1}^{\infty} P_{00}^{n}$ is finite or infinite (Theorem 3).

One can show that

$$\sum_{n=1}^{\infty} P_{00}^{(n)} = \infty \text{ if and only if } p = \tfrac{1}{2}$$

(See Ross, Example 4.18, page 199, 11th Edition)

So, the MC (or random walk) is recurrent when $p = \frac{1}{2}$, and transient if $p \neq \frac{1}{2}$. //