Classification of States

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Suppose that $\{X_n \mid n \geq 0\}$ is a Markov chain with transition probability matrix, $\mathbb{P} = (P_{ij})$,

where

$$P_{ij} = P\{X_{n+1} = j \mid X_n = i\}$$
$$= P\{X_1 = j \mid X_0 = i\},$$

is called one-step conditional probability, defined, for all $n \geq 0$.

Let $P_{ij}^{(n)}$ be the n-step transition probability defined by

$$P_{ij}^{(n)} = P\{X_{n+k} = j / X_k = i\}$$
$$= P\{X_n = j / X_0 = i\}$$

where

$$P_{ij}^{(1)} = P_{ij},$$
$$P_{ii}^{(0)} = 1,$$
$$P_{ij}^{(0)} = 0, \quad i \neq j,$$
$$\forall n \geq 0, \quad i, j \geq 0.$$

Definition 1: We say that state j is accessible (or reachable) from state i, denoted as $i \to j$, if $P_{ij}^{(n)} > 0$, for some integer $n \geq 0$.

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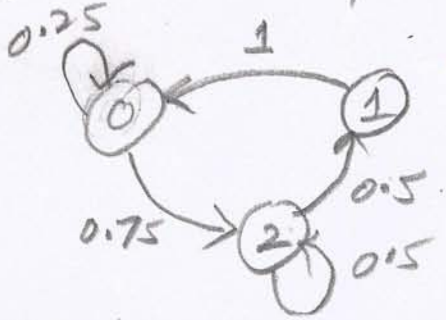
Example 44. Consider a 3-state Markov chain with transition probability matrix given by

$$P = \begin{array}{c} 0 \\ 1 \\ 2 \end{array} \left\| \begin{array}{ccc} 0.25 & 0 & 0.75 \\ 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \end{array} \right\|$$

Show the accessibility between the all pair of states, if exists.

Solution:



(The transition graph of given Markov chain)

The following table shows all possible pair of states which are accessible from each others:

$i \to j$	Path	$P_{ij}^{(n)},\ n \geq 0$
$0 \to 1$	0, 2, 1	$P_{01}^{(2)} = (0.75)(0.5) = 0.375 > 0$
$1 \to 0$	1, 0	$P_{10} = P_{10}^{(1)} = 1 > 0$
$1 \to 2$	1, 0, 2	$P_{12}^{(2)} = (1)(0.75) = 0.75 > 0$
$2 \to 1$	2, 1	$P_{21}^{(1)} = 0.5 > 0$
$0 \to 2$	0, 2	$P_{02}^{(1)} = 0.75 > 0$
$2 \to 0$	2, 1, 0	$P_{20}^{(2)} = (0.5)(1) = 0.5 > 0$

Definition 2. We say that states i and j Communicate, denoted by $i \leftrightarrow j$, if both j is accessible from i, and i is accessible from j.

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Define a relation 'R' on the set of states, S, by

$$i \, R \, j \text{ if, and only if } i \leftrightarrow j$$

It can be shown that the relation R is an equivalence relation.

Proof:

(a) (Reflexivity) $i \leftrightarrow i \quad \forall i \in S$

Since, $\exists \, n = 0$, such that

$$P_{ii}^{(0)} = 1 > 0 \quad \forall i \in S$$

(b) (Symmetric).

Since, by definition,

$$i \leftrightarrow j \iff j \leftrightarrow i \quad \forall i, j \in S$$

(c) (Transitivity)

We have, $(i \leftrightarrow j)$ and $(j \leftrightarrow k) \Rightarrow (i \rightarrow k) \quad \forall i, j, k \in S$

Since, $(i \leftrightarrow j)$ and $(j \leftrightarrow k)$ implies that there are integers $n, m \geq 0$ such that

$$P_{ij}^{(n)} > 0 \text{ and } P_{jk}^{(m)} > 0.$$

By the Chapman-Kolmogorov equation, we have

$$P_{ik}^{(n+m)} = \sum_{r=0}^{\infty} P_{ir}^{(n)} P_{rk}^{(m)}$$

$$> P_{ij}^{(n)} P_{jk}^{(m)} > 0.$$

$\Rightarrow (i \leftrightarrow k)$. //

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The relation R, of communication divides the state space, S, into a number of separate <u>equivalence classes</u>*.

(An <u>equivalence class</u> consists of all states that communicate with each other).

Consider the 3-state Markov chain discussed in Example 44. It consists of <u>only</u> <u>one</u> <u>class</u>, viz., $\{0, 1, 2\}$.

Since, $0 \leftrightarrow 1$, $1 \leftrightarrow 2$, $0 \leftrightarrow 2$.

Also, $P_{00}^{(1)} = \underset{>0}{0.25}$, $P_{22}^{(1)} = 0.5 > 0$

and
$$P_{11}^{(3)} = P_{10} P_{02} P_{21}$$
$$= (1)(0.75)(0.5)$$
$$= 0.375 > 0$$

<u>Definition 3</u>. We say that the Markov chain is <u>irreducible</u> if there is only one class, that is, if all states communicate with each other.

Example 4.5. Consider a 4-state Markov chain having transition probability matrix,

$$P = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \left\| \begin{array}{cccc} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{array} \right\|$$

Find the equivalence classes of state space, $\{0, 1, 2, 3\}$. Is it irreducible?

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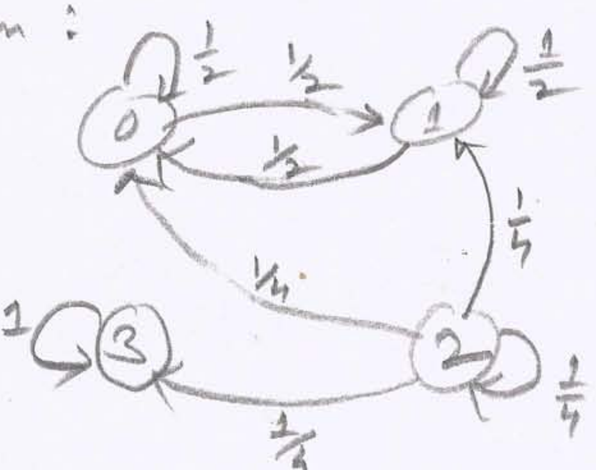
Solution. We first draw the transition diagram:



From the transition diagram, we can observe that

(i) $0 \Leftrightarrow 1$, since $P_{01} = \frac{1}{2} > 0$, and $P_{10} = \frac{1}{2} > 0$.

(ii) $2 \to 0$, $2 \to 1$, $2 \to 2$, $2 \to 3$, since $P_{2j} = \frac{1}{4} > 0 \;\forall j = 0, 1, 2, 3$

(iii) $3 \Leftrightarrow 3$, since $P_{33} = 1 > 0$.

Continued...

But,

(iv) $0 \not\rightarrow 2$, $1 \not\rightarrow 2$, $3 \not\rightarrow 2$,

since $P_{02} = P_{12} = P_{32} = 0$,

and (v) $3 \not\rightarrow 0$, $3 \not\rightarrow 1$, $3 \not\rightarrow 2$,

since $P_{30} = P_{31} = P_{32} = 0$.

Therefore, the <u>equivalence classes</u> of this markov chain are

$\{0, 1\}$, $\{2\}$, and $\{3\}$

respectively.

Clearly, this Markov chain is

<u>not</u> irreducible. //

<u>Note:</u> The <u>state 3</u> in the above example 45, is the <u>absorbing state</u>, since once entered, it is never left, in other words, no other state is accessible from it.

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