Markov Chains - Continued ...

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Dr. SUSHIL KUMAR AZAD

In the last lecture, we defined the one-step transition probabilities, P_{ij}.

Consider the Example 38 (Forecasting the weather) again:

Problem - Suppose we wish to find the probability that it will rain four days from today given that it is raining today, that is, the probability that a process in a current state i (it is raining today) will be in state j (it will rain) after 4-additional transition (after 4 days from today).

Solution. Let us denote the probability that a process in state i & will be in state j after n additional transitions, by P_{ij}^n.

Then,
$$P_{ij}^n = P\{X_{n+k} = j \mid X_k = i\}, \quad n \geq 0, \; i, j \geq 0$$

Clearly, $P_{ij}^1 = p_{ij}$

In the above example, we desire the probability, P_{00}^4.

A method for computing P_{ij}^n, called the n-step transition probabilities is provided by Chapman_Kolmogorov equations given by

$$P_{ij}^{n+m} = \sum_{k=0}^{\infty} P_{ik}^n P_{kj}^m \quad \text{...(i)}$$

for all $n, m \geq 0$, and all i, j

— .. —

<u>Proof</u> (of Chapman–Kolmogorov equations)

$$i \xrightarrow{n\text{-steps}} k \xrightarrow{m\text{-steps}} j$$

The probability that starting in i, the process will go to state j in $(n+m)$-transitions is given by

$$P_{ij}^{n+m} = P\{X_{n+m}=j \mid X_0=i\} \quad \text{---(ii)}$$

whereas the term $P_{ik}^{n} P_{kj}^{m}$ on the R.H.S. of (i), represents the probability that starting in i, the process will go to state j in $n+m$ transitions through a path which takes into state k at the nth transition, as shown in above figure.

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From (ii), we get

$$P_{ij}^{n+m} = \sum_{k=0}^{\infty} P\{X_{n+m}=j,\ X_n=k \mid X_0=i\}$$

$$= \sum_{k=0}^{\infty} P\{X_{n+m}=j \mid X_n=k, X_0=i\} \times P\{X_n=k \mid X_0=i\}, \text{ as } P[A\cap B|C] = P[A|B\cap C]\,P[B|C]$$

$$= \sum_{k=0}^{\infty} P\{X_{n+m}=j \mid X_n=k\} \times P\{X_n=k \mid X_0=i\}$$
(by Markov property)

$$= \sum_{k=0}^{\infty} P_{kj}^{m} P_{ik}^{n},$$ Hence the result ||.

Let us now consider the Example 3.5 in which we wish to calculate P_{00}^{4}, i.e. the probability that it will rain 4-days from today, given that it is raining today.

From Example 39, we have the one-step transition probability matrix for the 2-state Markov Chain $\{X_n \mid n = 0, 1\}$, where X_0 represents the state 0 when it rains and X_1 represents the state 1 when it does not rain, is given by

$$P = \begin{Vmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{Vmatrix} \quad \text{—— (iii)}$$

We have, $p_{ij} = P^{1}_{ij}$.

Let $P^{(n)}$ denote the matrix of n-step transition probabilities $P^{(n)}_{ij}$.

Then, the Chapman-Kolmogorov equations implies that $P^{(n+m)} = P^{(n)} \cdot P^{(m)}$.

In particular, $P^{(2)} = P^{(1+1)} = P^1 P^1$

$$\Rightarrow P^{(2)} = P \cdot P = P^2$$

Hence, by induction we have

$$P^{(n)} = P^{(n-1+1)} = P^{n-1} P^1 = P^n$$

Now, from (iii), we get

$$P^{(2)} = P^2 = \begin{Vmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{Vmatrix} \begin{Vmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{Vmatrix}$$

$$= \begin{Vmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{Vmatrix},$$

and

$$P^{(4)} = P^{(2+2)} = P^2 P^2 = \begin{Vmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{Vmatrix} \begin{Vmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{Vmatrix}$$

$$= \begin{Vmatrix} 0.5749 & 0.4251 \\ 0.5668 & 0.4332 \end{Vmatrix}.$$

Hence, the required probability,

$$P_{00}^{4} = 0.5749$$

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Note 1. Recall that a sequence of discrete r.v.s $\{X_n \mid n = 1, 2, \cdots\}$ is a Markov Chain if it satisfies the Markov property:

$$\boxed{P\{X_{n+1} = j \mid X_n = k, \cdots, X_0 = i\} = P\{X_{n+1} = j \mid X_n = k\}}$$

for all $n \geq 1$ & for all states $x_0, x_1, \cdots$

2. Further,

$$P\{X_1 = j / X_0 = i\} = P\{X_{n+1} = j / X_n = i\} \text{ for any } n.$$
$$= p_{ij}$$

p_{ij} is the probability of making a transition from state i to state j in a single step.

3. Consider the probability of making a transition from state i to state j over two steps:

$$P\{X_2 = j \mid X_0 = i\}$$
$$= \sum_{k=1}^{N} P\{X_2 = j \mid X_1 = k, X_0 = i\} * P\{X_1 = k \mid X_0 = i\},$$

(assuming the state space $S = \{1, 2, \cdots, N\}$)

as $P[A|C] = P[A|B \cap C] * P[B|C]$

$$\therefore P\{X_2 = j / X_0 = i\} = \sum_{k=1}^{N} P\{X_2 = j / X_1 = k\} * P\{X_1 = k / X_0 = i\}$$

(Markov Property)

$$= \sum_{k=1}^{N} p_{kj}\, p_{ik} \quad \text{(by definitions)}$$

$$= \sum_{k=1}^{N} p_{ik}\, p_{kj} \quad \text{(rearranging)}$$

$$= (P^2)_{ij} \quad \text{(by definition)}$$

In general, we have

$$\boxed{P\{X_{n+m} = j / X_n = i\} = P\{X_m = j / X_0 = i\} = (P^m)_{ij} \text{ for any } n}$$

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Example 42. Let $\{X_t = t = 0, 1, 2, \ldots\}$ be a Markov chain having the transition matrix

$$P = \begin{Vmatrix} 0.6 & 0.2 & 0.2 \\ 0.4 & 0 & 0.6 \\ 0 & 0.8 & 0.2 \end{Vmatrix}$$

Find $P\{X_2 = 3 / X_0 = 1\}$

Solution. We have

$$P\{X_2 = 3 / X_0 = 1\} = (P^2)_{13} \qquad \text{(Note 3.)}$$

$$= \begin{Vmatrix} 0.6 & 0.2 & 0.2 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{Vmatrix} \begin{Vmatrix} \cdot & \cdot & 0.2 \\ \cdot & \cdot & 0.6 \\ \cdot & \cdot & 0.2 \end{Vmatrix}$$

$$= 0.6 \times 0.2 + 0.2 \times 0.6 + 0.2 \times 0.2$$

$$= 0.28 \quad //$$

Example 43. In sociology, it is convenient to classify people by income as lower-class, middle-class, and upper class, If an individual in the lower-income class is said to be in State 1, an individual in the middle-income class is in state 2, and an individual in the upper-income class is in State 3 then the probabilities p_{ij} of change in income class from one generation to the next is given as follows:

	State	Next Generation 1	2	3
Current Generation	1	0.65	0.28	0.07
	2	0.15	0.67	0.18
	3	0.12	0.36	0.52

e.g., $p_{23} = 0.18$, $p_{31} = 0.12$, $p_{22} = 0.67$ and so on.

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(6)

Write the transition matrix P for the said process. If a parent is in state 3, find the probability that a grandchild will be in state 2.

__Solution.__ The transition matrix P for the given 3-state Markov chain is (from the table above):

$$P = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0.65 & 0.28 & 0.07 \\ 2 & 0.15 & 0.67 & 0.18 \\ 3 & 0.12 & 0.36 & 0.52 \end{array}$$

In P, a person is assumed to be in one of three discrete states (lower, middle or upper income), with each offspring in one of these same three discrete states.

Next, to compute the probability that a grandchild will be in state 2, if parent is in state 3, consider the following tree diagram:

Current generation (parent)	Next generation (Child)	Third Generation (Grand Child)	Probabilities
3	$\xrightarrow{0.12}$ 1	$\xrightarrow{0.65}$ 1	$(0.12)(0.65)=0.078$
		$\xrightarrow{0.28}$ 2	$\boxed{(0.12)(0.28)=0.0336}$
		$\xrightarrow{0.07}$ 3	$(0.12)(0.07)=0.0084$
	$\xrightarrow{0.36}$ 2	$\xrightarrow{0.15}$ 1	$(0.36)(0.15)=0.054$
		$\xrightarrow{0.67}$ 2	$\boxed{0.36\times0.67=0.2412}$
		$\xrightarrow{0.18}$ 3	$(0.36)(0.18)=0.0634$
	$\xrightarrow{0.52}$ 3	$\xrightarrow{0.12}$ 1	$(0.52)\times(0.12)=0.0624$
		$\xrightarrow{0.36}$ 2	$\boxed{(0.52)(0.36)=0.1872}$
		$\xrightarrow{0.52}$ 3	$(0.52)(0.52)=0.2704$

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From the tree diagram, we can observe that the probability that a parent is in state 3 will have a grandchild in state 2 is given by the sum of the probabilities indicated in closed blocks, that is,

$$= p_{31} \cdot p_{12} + p_{32} \cdot p_{22} + p_{33} \cdot p_{32} \quad \text{---(iv)}$$

$$= 0.0336 + 0.2412 + 0.1872$$

$$= 0.4620$$

Note that, the expression (iv) is $(P^2)_{32}$

$$= \left\| \begin{matrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0.12 & 0.36 & 0.52 \end{matrix} \right\| \left\| \begin{matrix} \cdot & 0.28 & \cdot \\ \cdot & 0.67 & \cdot \\ \cdot & 0.36 & \cdot \end{matrix} \right\|$$

Hence, the probability that a parent is in state 3 will have grandchild in state 2 is $= (P^2)_{32} = 0.4620$

using the 2-state transition matrix, P^2.

(see Note 2, page 4)

Sushil Kumar Arora