March 31, 2020

Markov Chains

Dr. SUSHIL KUMAR AZAD

In the last lectures, we have discussed two of the principal theorems for independent trials processes: the Law of Large Numbers and the Central Limit Theorem.

When a sequence of chance experiments forms an independent trials process, the possible outcomes for each experiment are the same and occur with the same probability. In these processes, the Knowledge of the outcomes of the previous experiments does not influence our predictions for the outcomes of the next experiment.

In Modern probability theory, we study the chance processes for which the Knowledge of previous outcomes influences predictions for future experiments.

In 1907, A. A. Markov (1856-1922), Russian mathematician, studied the chance process in which the outcome of a given experiment can affect the outcome of the next experiment. This type of process is called a Markov Chain.

To understand it, let us consider the following example of forecasting the weather:

Example 39. Suppose that the chance of rain tomorrow depends on previous weather conditions whether or not it is raining today and not on past weather conditions.

Continued

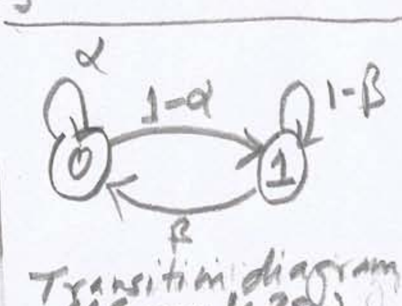
Clearly, for the example above, we have two states, viz., the process is in <u>state 0</u> when it rains and <u>state 1</u>, when it does not rain. Let X_n, $n=0,1$ be the state of the process at time n, i.e. $X_0 \equiv X(0)$ represents the state 0, and $X_1 \equiv X(1)$, represents the state 1.

Thus, we have a two-state process $\{X_n \mid n=0,1\}$.

Suppose that if it rains today, then it will rain tomorrow with probability α, and if it does not rain today, then it will rain tomorrow with probability β.

Let p_{ij}, represents the probability that the process will, when in state i, next make a transition into state j.

Then
$$p_{00} = P\{X_1 = 0 \mid X_0 = 0\} = \alpha,$$
$$p_{01} = P\{X_1 = 1 \mid X_0 = 0\} = 1-\alpha,$$
$$p_{10} = P\{X_1 = 0 \mid X_0 = 1\} = \beta, \text{ and}$$
$$p_{11} = P\{X_1 = 1 \mid X_0 = 1\} = 1-\beta$$



Transition diagram ((Example 35))

These transition probabilities can be expressed in matrix form, P, given by

$$P = \begin{Vmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{Vmatrix} = \begin{Vmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{Vmatrix}.$$

SUSHIL KUMAR AZAD

<u>Definition 1.</u> A (finite) <u>Markov Chain</u> is a process with a finite number of states (or outcomes, or events) in which the probability of being in a particular state at step $n+1$ depends only on the state occupied at step n.

SUSHIL KUMAR AZAD

Let $S = \{S_0, S_1, \ldots S_r\}$ be the possible states.

Let
$$\vec{P}_n = \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_r \end{bmatrix},$$
be the vector of probabilities of each state at step n, i.e., the i^{th} entry of $\vec{P}_n$, $\vec{P}_n[i]$, is the probability that the process is in state S_i at step n.

For such a probability vector, we have
$$p_0 + p_1 + p_2 + \cdots + p_r = 1$$

Let $p_{ij} = P[\text{State } n+1 = S_j \mid \text{State } n = S_i]$

and let
$$P = \begin{bmatrix} p_{00} & p_{01} & \cdots & p_{0r} \\ p_{10} & p_{11} & \cdots & p_{1r} \\ \vdots & & \ddots & \\ p_{r0} & p_{r1} & \cdots & p_{rr} \end{bmatrix}$$

That is, p_{ij} is the Conditional probability of being in state S_j at step $n+1$, given that the process was in state S_i at step n.

P is called the <u>transition matrix</u>.

We have $p_{ij} \geq 0$, $i, j \geq 0$;
$$\sum_{j=0}^{r} p_{ij} = 1, \quad i = 0, 1, \ldots, r$$

A markov chain is a special kind of stochastic process which is a mathematical model that evolves over time in a probabilistic manner.

A stochastic process $\{X(t) \mid t \in T\}$ is a collection of random variables, i.e. for each $t \in T$, $X(t)$ is a random variable. The index t, is often interpreted as time and, $X(t)$ as the state of the process at time t. The set T is called the index set of the process. When T is countable set, the stochastic process is said to be a discrete time process.

If T is an interval of the real line, the stochastic process is said to be a continuous-time process.

e.g. $\{X_n \mid n = 0, 1, \dots\}$ is a discrete-time stochastic process indexed by the nonnegative integers.

Sushil Kumar Azad

Definition 2 A stochastic process $\{X_n \mid n = 0, 1, \dots\}$ in discrete time with finite or infinite state space S is a Markov chain with stationary transition probabilities if it satisfies:

(1) For each $n \geq 1$, if A is an event depending only on any subset of $\{X_{n-1}, X_{n-2}, \dots, X_0\}$, then for any states i and j in S,

$$P[X_{n+1} = j \mid X_n = i \text{ and } A] = P[X_{n+1} = j \mid X_n = i]$$

Kumar Azad

Sushil

(2) For any given states i and j

$$P[X_{n+1} = j \mid X_n = i] \text{ is same } \forall\, n \geq 0$$

In simple words, a sequence of trials of an experiment is a Markov Chain if

(1) the outcome of each experiment is one of a set of discrete states;

(2) the outcome of an experiment depends only on the present state, and not on any past states.

Consider the following gambler's ruin problem:

Example 40: Suppose that a gambler starts playing a game with an initial $\$B$ bank roll. The game proceeds in turns, where at the end of each turn the gambler either wins $\$1$ with probability p, or loses $\$1$ with probability $q = 1-p$. The gambler continues until he or she either makes it to $\$N$, or goes bankrupt with $\$0$. Determine the transition matrix, when the gambler's ruin with $N=4$.

Solution. We can represent the gambler's ruin process by a Markov Chain having $N+1$ states representing the amount of money that the player has: either $\$0, \$1, \cdots,$ or $\$N$.

that is, let X_n denote the player's fortune at time n. Then the process $\{X_n \mid n = 0, 1, 2, \dots, N\}$ is a Markov Chain with transition probabilities given as follows (gambler quits playing when he goes broke or he attains a fortune of \$N):

$$P_{00} = 1;$$
$$P_{NN} = 1; \text{ and}$$

$$P_{i,i+1} = p \text{ and}$$
$$P_{i,i-1} = q \quad \text{for } i = 1, 2, \dots, N-1$$

The corresponding transition matrix is,

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ \\ N-2 \\ N-1 \\ N \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & \cdots & N-1 & N \end{array} \\ \left[\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ q & 0 & p & 0 & 0 & \cdots & 0 & 0 \\ 0 & q & 0 & p & 0 & \cdots & 0 & 0 \\ 0 & 0 & q & 0 & p & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & p & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & p \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & & \end{array}\right] \end{array}$$

For $N = 4$, we have,

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 \\ 0 & q & 0 & p & 0 \\ 0 & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \end{array}\right] \end{array},$$

where,

$$p_{ij} = P[X_{n+1} = j \mid X_n = i], \text{ i.e.}$$
$$p_{01} = P[X_{n+1} = 1 \mid X_n = 0] = p, \text{ and}$$
$$p_{10} = P[X_{n+1} = 0 \mid X_n = 1] = q \text{ etc.}$$

Example 41. Suppose that whether or not it rains today depends on the previous weather conditions through the last 2 days. Suppose that

(i) if it has rained for the past 2 days, then it will rain tomorrow with probability 0.7 ;

(ii) if it rained today but not yesterday, then it will rain tomorrow with probability 0.5;

(iii) if it rained yesterday but not today, then it will rain tomorrow with probability 0.4; and

(iv) if it has not rained in the past 2 days, then it will rain tomorrow with probability 0.2.

Write a Markov chain for the process and also write the one step transition probability matrix.

Solution. Let X_n be the random variable that denotes the weather conditions on nth day and $(n-1)^{th}$ day. Then X_n takes values 0, 1, 2 or 3 according to

$$X_n = \begin{cases} 0, & \text{case (i)} \\ 1, & \text{case (ii)} \\ 2, & \text{case (iii)} \\ 3, & \text{case (iv)} \end{cases}$$

Then $\{X_n \mid n = 0, 1, 2, 3\}$ represent a four-state Markov chain, with the transition probability matrix given as follows:

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{array} \right\| \end{array}$$

The transition diagram of this process is