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Dr. SUSHIL KUMAR AZAD

Conditional Distributions and Expectations

In this section, we shall discuss conditional distributions, i.e. the distribution of one of the random variables when the other has assumed a specific value.

(i) Discrete Case:-

Let X_1 and X_2 be the random variables having the joint pmf, $p_{X_1,X_2}(x_1, x_2)$, that is positive on the support set S and is zero elsewhere.

We know, the marginal pmfs of X_1 and X_2 are given by

$$p_{X_1}(x_1) = \sum_{x_2 < \infty} p_{X_1,X_2}(x_1, x_2), \text{ for all } x_1 \in S_{X_1}, \text{ the support of } X_1$$

and

$$p_{X_2}(x_2) = \sum_{x_1 < \infty} p_{X_1,X_2}(x_1, x_2), \text{ for all } x_2 \in S_{X_2}, \text{ the support of } X_2$$

respectively.

Let $x_1 \in D_{X_1}$. Then, using the definition of conditional probability, we have

$$P(X_2 = x_2 \mid X_1 = x_1) = \frac{P(X_1 = x_1, X_2 = x_2)}{P(X_1 = x_1)}$$

$$= \frac{p_{X_1,X_2}(x_1, x_2)}{p_{X_1}(x_1)}$$

for all $x_2 \in S_{X_2}$ of X_2.

We denote this function by $p_{X_2|X_1}(x_2|x_1)$, called the <u>conditional pmf</u> of X_2, given that $X_1 = x_1$, that is,

$$p_{X_2|X_1}(x_2|x_1) = \frac{p_{X_1,X_2}(x_1, x_2)}{p_{X_1}(x_1)}, \quad x_2 \in S_{X_2}.$$

Note that

$$\sum_{x_2 \in S_{X_2}} p_{X_2|X_1}(x_2|x_1) = \sum_{x_2 \in S_{X_2}} \frac{p_{X_1,X_2}(x_1,x_2)}{p_{X_1}(x_1)}$$

$$= \frac{1}{p_{X_1}(x_1)} \sum_{x_2} p_{X_1,X_2}(x_1,x_2)$$

$$= \frac{p_{X_1}(x_1)}{p_{X_1}(x_1)} = 1.$$

In a similar manner, provided $x_2 \in S_{X_2}$, we define the conditional pmf of X_1, given that $X_2 = x_2$, by

$$p_{X_1|X_2}(x_1|x_2) = \frac{p_{X_1,X_2}(x_1,x_2)}{p_{X_2}(x_2)}, \quad x_1 \in S_{X_1}$$

<u>Note</u> We often abbreviate $p_{X_1|X_2}(x_1|x_2)$ by $p_{1|2}(x_1|x_2)$, $p_{X_2|X_1}(x_2|x_1)$ by $p_{2|1}(x_2|x_1)$, $p_{X_1}(x_1)$ by $p_1(x_1)$ and $p_{X_2}(x_2)$ by $p_2(x_2)$.

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(ii) <u>Continuous Case</u> :-

Now let X_1 and X_2 be the random variables having the joint pdf $f_{X_1,X_2}(x_1,x_2)$. We know that the marginal pdfs $f_{X_1}(x_1)$ and $f_{X_2}(x_2)$ are given by

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)\, dx_2 \quad \forall x_1 \in S_{X_1}$$

and

$$f_{X_2}(x_2) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)\, dx_1 \quad \forall x_2 \in S_{X_2}$$

Let $x_1 \in S_{X_1}$. We define the <u>conditional pdf</u> of X_2, given that $X_1 = x_1$, by

$$f_{X_2|X_1}(x_2|x_1) = \frac{f_{X_1,X_2}(x_1,x_2)}{f_{X_1}(x_1)}$$

Note that

$$f_{X_2|X_1}(x_2|x_1) \geq 0 \text{ and}$$

$$\int_{-\infty}^{\infty} f_{X_2|X_1}(x_2|x_1)\,dx_2 = \frac{1}{f_{X_1}(x_1)} \int_{-\infty}^{\infty} f_{X_1 X_2}(x_1, x_2)\,dx_2$$

$$= \frac{1}{f_{X_1}(x_1)} \times f_{X_1}(x_1) = 1.$$

that is, $f_{X_2|X_1}(x_2|x_1)$ has the properties of a pdf of one continuous type of random variable.

In a similar manner, when $x_2 \in S_{X_2}$, the <u>conditional pdf</u> of X_1, given that $X_2 = x_2$, is

$$f_{X_1|X_2}(x_1|x_2) = \frac{f_{X_1,X_2}(x_1,x_2)}{f_{X_2}(x_2)}$$

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We often abbreviate $f_{X_2|X_1}(x_2|x_1)$ by $f_{2|1}(x_2|x_1)$, $f_{X_1|X_2}(x_1|x_2)$ by $f_{1|2}(x_1, x_2)$, $f_{X_1}(x_1)$ by $f_1(x_1)$ and $f_{X_2}(x_2)$ by $f_2(x_2)$.

Thus, "the <u>conditional probability</u> that $a < X_2 < b$, given that $X_1 = x_1$", is given by

$$P(a < X_2 < b \mid X_1 = x_1) = \int_a^b f_{2|1}(x_2|x_1)\,dx_2$$

Conditional Expectation

If $g(X_2)$ is a function of X_2, we define the <u>conditional expectation</u> of $g(X_2)$, given that $X_1 = x_1$ by

$$E[g(X_2)|x_1] = \int_{-\infty}^{\infty} g(x_2) f_{2|1}(x_2|x_1)\,dx_2,$$

if $\int_{-\infty}^{\infty} |g(x_2)| f_{2|1}(x_2|x_1)\,dx_2 < \infty$

Note that, $E[g(X_2)|x_1]$ is a function of x_1, and $E(X_2|x_1)$ is the mean of the conditional distribution of X_2, given that $X_1 = x_1$.

We define the <u>conditional variance</u> of the conditional distribution of X_2, given $X_1 = x_1$, written as $Var(X_2|x_1)$, by

$$Var(X_2|x_1) = E\left([X_2 - E(X_2|x_1)]^2 \,|\, x_1\right)$$

which is equivalent to

$$var(X_2|x_1) = E(X_2^2|x_1) - [E(X_2|x_1)]^2$$

In a similar manner, the conditional expectation of $g(X_1)$, given $X_2 = x_2$, is given by

$$E[g(X_1)|x_2] = \int_{-\infty}^{\infty} g(x_1) f_{1|2}(x_1|x_2)\,dx_1,$$

if $\int_{-\infty}^{\infty} |g(x_1)| f_{1|2}(x_1|x_2)\,dx_1 < \infty$.

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Example 4: Let X_1 and X_2 have the joint pmf $p(x_1, x_2)$ described as follows:

(x_1, x_2)	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$	$(2,0)$	$(2,1)$
$p(x_1, x_2)$	$\frac{1}{18}$	$\frac{3}{18}$	$\frac{4}{18}$	$\frac{3}{18}$	$\frac{6}{18}$	$\frac{1}{18}$

and $p(x_1, x_2)$ is zero elsewhere. Find the two marginal pmfs and the two conditional means.

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Solution: Writing the joint pmf $p(x_1, x_2)$ in an rectangular array, we have

		Support of X_2		
	x_1	0	1	$p_1(x_1)$
Support of X_1	0	$\frac{1}{18}$	$\frac{3}{18}$	$\frac{4}{18}$
	1	$4/18$	$3/18$	$7/18$
	2	$\frac{6}{18}$	$1/18$	$7/18$
	$p_2(x_2)$	$\frac{11}{18}$	$\frac{7}{18}$	1

Thus, the marginal pmf of X_1, $p_1(x_1)$, is

x_1	0	1	2
$p_1(x_1)$	$\frac{4}{18}$	$\frac{7}{18}$	$\frac{7}{18}$

and, = zero elsewhere, and

the marginal pmf of X_2, $p_2(x_2)$, is

x_2	0	1
$p_2(x_2)$	$\frac{11}{18}$	$\frac{7}{18}$

and is = zero, elsewhere

Cont...

Next, the conditional pmf of X_2, given that $X_1 = x_1$,

$$p_{2|1}(x_2|x_1) = \frac{p(x_1, x_2)}{p_1(x_1)}, \quad x_2 \in S_{X_2},$$

is given by

x_2	0	1
$p_{2\|1}(x_2\|0)$	$\left(\frac{1}{18}\right)/\left(\frac{4}{18}\right) = \frac{1}{4}$	$\left(\frac{3}{18}\right)/\left(\frac{4}{18}\right) = \frac{3}{4}$
$p_{2\|1}(x_2\|1)$	$\left(\frac{4}{18}\right)/\left(\frac{7}{18}\right) = \frac{4}{7}$	$\left(\frac{3}{18}\right)/\left(\frac{7}{18}\right) = \frac{3}{7}$,
$p_{2\|1}(x_2\|2)$	$\left(\frac{6}{18}\right)/\left(\frac{7}{18}\right) = \frac{6}{7}$	$\left(\frac{1}{18}\right)/\left(\frac{7}{18}\right) = \frac{1}{7}$

and the conditional pmf of X_1, given that $X_2 = x_2$,

$$P_{1|2}(x_1|x_2) = \frac{p(x_1, x_2)}{p_2(x_2)}, \text{ is given by.}$$

x_1	$p_{1\|2}(x_1\|0)$	$p_{1\|2}(x_1\|1)$
0	$\left(\frac{1}{18}\right)/\left(\frac{11}{18}\right) = \frac{1}{11}$	$\left(\frac{3}{18}\right)/\left(\frac{7}{18}\right) = \frac{3}{7}$
1	$\left(\frac{4}{18}\right)/\left(\frac{11}{18}\right) = \frac{4}{11}$	$\left(\frac{3}{18}\right)/\left(\frac{7}{18}\right) = \frac{3}{7}$
2	$\left(\frac{6}{18}\right)/\left(\frac{11}{18}\right) = \frac{6}{11}$	$\left(\frac{1}{18}\right)/\left(\frac{7}{18}\right) = \frac{1}{7}$

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Now, the conditional mean of marginal distribution of X_2, given $X_1 = x_1$, $E(X_2|x_1)$ is given by.

$$E(X_2|x_1) = \sum_{x_2 \in S_{X_2}} x_2\, p_{2|1}(x_2|x_1)$$

Thus,

x_1	0	1	2
$E(X_2\|x_1)$	$\frac{3}{4}$	$\frac{3}{7}$	$\frac{1}{7}$

where

$$E(X_2|0) = \sum_{x_2} x_2\, p_{2|1}(x_2|0)$$

$$= 0 \times \frac{1}{4} + 1 \times \frac{3}{4} = \frac{3}{4},$$

$$E(X_2|1) = \sum_{x_2} x_2\, p_{2|1}(x_2|1)$$

$$= 0 \times \frac{4}{7} + 1 \times \frac{3}{7} = \frac{3}{7}, \text{ and}$$

$$E(X_2|2) = \sum_{x_2} x_2\, p_{2|1}(x_2|2)$$

$$= 0 \times \frac{6}{7} + 1 \times \frac{1}{7} = \frac{1}{7}.$$

In a similar manner, the conditional mean of X_1, given $X_2 = x_2$, $E(X_1|x_2)$ is

x_2	0	1
$E(X_1/x_2)$	$\frac{16}{11}$	$\frac{5}{7}$

where $E(X_1|0) = \sum_{x_1} x_1\, p_{1|2}(x_1|0) =$

$$= 0 \times \frac{1}{11} + 1 \times \frac{4}{11} + 2 \times \frac{6}{11} = \frac{16}{11}, \text{ and}$$

$$E(X_1|1) = 0 \times 3/7 + 1 \times \frac{3}{7} + 2 \times \frac{1}{7} = \frac{5}{7}$$

Example 5: Let $f(x_1, x_2) = \begin{cases} 21\, x_1^2 x_2^3, & 0 < x_1 < x_2 < 1 \\ 0 & \text{elsewhere} \end{cases}$

be the joint pdf of X_1 and X_2.

(a) Find the conditional mean and variance of X_1, given $X_2 = x_2$, $0 < x_2 < 1$.

(b) Find the distribution of $Y = E(X_1 | X_2)$.

Solution: (a) The marginal pdf of X_2 is

$$f_2(x_2) = \int_0^{x_2} 21\, x_1^2 x_2^3\, dx_1 = \begin{cases} 7 x_2^6, & 0 < x_2 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

The conditional pdf of X_1, given $X_2 = x_2$, is

$$f_{1|2}(x_1|x_2) = \frac{21 x_1^2 x_2^3}{7 x_2^6} = \frac{3 x_1^2}{x_2^3}, \quad 0 < x_1 < x_2$$

Thus, the conditional mean of X_1 (given $X_2 = x_2$), $E(X_1|x_2)$, is

$$E(X_1|x_2) = \int_0^{x_2} x_1 \cdot f_{1|2}(x_1|x_2)\, dx_1$$

$$= \int_0^{x_2} 3\, x_1^3\, x_2^{-3}\, dx_1$$

$$= 3 x_2^{-3} \left| \frac{x_1^4}{4} \right|_0^{x_2} = \frac{3 x_2}{4}, \quad 0 < x_2 < 1.$$

and the variance of X_1, given $X_2 = x_2$, is

$$\mathrm{var}(X_1|x_2) = \int_0^{x_2} \left(x_1 - E(X_1|x_2)\right)^2 f_{1|2}(x_1|x_2)\, dx_1$$

$$= \int_0^{x_2} \left(x_1 - \frac{3x_2}{4}\right)^2 \frac{3 x_1^2}{x_2^3}\, dx_1$$

$$= \cdots = \frac{3}{80} x_2^2$$

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(b) Now, $Y = E(X_1|X_2) = \frac{3X_2}{4}$, $0 < X_2 < 1$.

So, the cdf of Y, $G(y)$, is

$$G(y) = P(Y \le y) = P\left(X_2 \le \frac{4y}{3}\right), \quad 0 \le y < \frac{3}{4}$$

$$= \int_0^{4y/3} f_2(x_2)\,dx_2$$

$$= \int_0^{4y/3} 7x_2^6\,dx_2 = \frac{7x_2^7}{7}\Big|_0^{4y/3} = \left(\frac{4}{3}\right)^7 y^7, \quad 0 \le y < \frac{3}{4}$$

and, hence the pdf of Y, $g(y)$, is

$$g(y) = \frac{d}{dy}(G(y)) = \begin{cases} 7\left(\frac{4}{3}\right)^7 y^6, & 0 < y < \frac{3}{4} \\ 0 & \text{elsewhere} \end{cases}$$

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Example 6: Let X and Y have the joint pdf

$$f(x, y) = \begin{cases} 2\exp\{-(x+y)\}, & 0 < x < y < \infty \\ 0 & \text{elsewhere} \end{cases}$$

Find the conditional mean $E(Y|x)$ of Y, given $X = x$.

Solution: The marginal pdf of X is

$$f_X(x) = 2\int_x^\infty e^{-x} e^{-y}\,dy = 2e^{-2x}, \quad 0 < x < \infty.$$

Hence, the conditional pdf of Y, given $X = x$, is

$$f_{Y/X}(y/x) = \frac{f(x, y)}{f_X(x)} = \frac{2e^{-x}e^{-y}}{2e^{-2x}}$$

$$= e^{-(y-x)}, \quad 0 < x < y < \infty$$

Contd ...

Now, the conditional mean $E(Y|X=x)$, is

$$E(Y|X=x) = \int_x^\infty y\, f_{2|1}(y|x)\, dy$$

$$= \int_x^\infty y\, e^{-(y-x)}\, dy$$

$$= (x+1), \quad x>0$$

Example 7 : Let the random variables X and Y have the joint density function

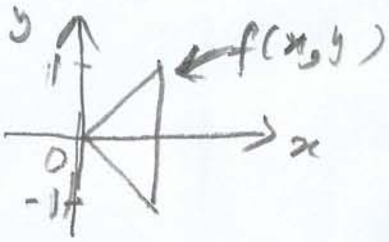
$$f(x,y) = \begin{cases} 1 & \text{if } -x<y<x,\ 0<x<1 \\ 0 & \text{elsewhere.} \end{cases}$$

Show that, on the set of positive probability density, the graph of $E(Y/x)$ is a straight line, whereas that of $E(X/y)$ is not a straight line.

Solution: The marginal pdf of X and Y, are

$$f_X(x) = \int_{-x}^{x} f(x,y)\,dy = \int_{-x}^{x} 1\,dy = 2x$$

and $$f_Y(y) = \int_{|y|}^{1} 1\cdot dx = 1-|y|, \quad |y|<1.$$



Hence, $$f_{X/Y}(x/y) = \frac{f(x,y)}{f_Y(y)} = \frac{1}{1-|y|}, \quad 0<|y|<x<1$$

and $$f_{Y/X}(y/x) = \frac{1}{2x}, \quad 0<|y|<x<1.$$

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Now, the conditional mean of X given $Y = y$, is

$$E(X/y) = \int_{|y|}^{1} x\, f_{X/y}(x/y)\, dx$$

$$= \int_{|y|}^{1} x \cdot \frac{1}{1-|y|}\, dx$$

$$= \frac{1}{1-|y|} \cdot \left| \frac{x^2}{2} \right|_{|y|}^{1}$$

$$= \frac{1-|y|^2}{2(1-|y|)} = \frac{1+|y|}{2}, \quad |y| < 1,$$

$$\Rightarrow E(X/y) = \frac{1+|y|}{2}, \quad |y|<1,$$ which is not a straight line.

whereas,

$$E(Y/x) = \int_{-x}^{x} y\, f_{Y/x}(y/x)\, dy$$

$$= \int_{-x}^{x} \frac{y}{2x}\, dy = \frac{1}{2x} \int_{-x}^{x} y\, dy$$

$$= \frac{1}{2x} \times 0 = 0, \quad 0 < x < 1$$

$\Rightarrow E(Y/x) = 0$ is a straight line, for $0 < x < 1$.

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