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Bivariate Normal Distribution

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The random variables X and Y are said to have a bivariate normal distribution if their joint density function is given by

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \times \exp\left\{ -\frac{1}{2(1-\rho^2)} \times \left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} \right] \right\}$$

$$-\infty < x, y, \mu_x, \mu_y < \infty, \quad \sigma_x, \sigma_y > 0, \quad \text{---(i)}$$

$$-1 \le \rho \le 1,$$

where $\mu_x, \mu_y, \sigma_x, \sigma_y$ and ρ are parameters of the distribution.

We denote $(X, Y) \sim BVN(\mu_x, \mu_y, \sigma_x, \sigma_y, \rho)$

Clearly, $f(x, y) \ge 0$, and

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dx\,dy = 1$, as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dx\,dy = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[\frac{-1}{2(1-\rho^2)}(u^2 - 2\rho uv + v^2)\right] du\,dv,$$

$\left[\text{Let } u = \frac{x-\mu_x}{\sigma_x}, \; v = \frac{y-\mu_y}{\sigma_y}\right]$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2(1-\rho^2)}\left\{(u-\rho v)^2 + (1-\rho^2)v^2\right\}\right] du\,dv$$

$$= \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{t^2}{2}\right) dt\right] \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{v^2}{2}\right) dv\right]$$

$\left[\text{Let } t = \frac{u-\rho v}{\sqrt{1-\rho^2}}\right]$

$$= 1 \times 1 = 1$$

<u>Remarks</u> (1) If $\rho^2 = 1$, the pdf (i) is not defined. In this case, the relation between X and Y is exactly linear, i.e. $Y = a + bX$. The distribution is then called a <u>bivariate degenerate</u> (or singular) <u>normal distribution</u>.

(2). If $(X, Y) \sim BVN(\mu_x, \mu_y, \sigma_x, \sigma_y, \rho)$, then X and Y are independent if, and only if $\rho = 0$.

(Note that, in general, $\rho = 0$ does not imply independency of X and Y)

<u>Proof</u>. Let $\rho = 0$, then from (i), we get

$$f(x, y) = \frac{1}{\sigma_x\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right] \times \frac{1}{\sigma_y\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]$$

i.e., $f(x, y) = f_1(x)\, f_2(y)$, as

$$f_1(x) = \int_{-\infty}^{\infty} f(x, y)\, dy = \frac{1}{2\pi\sigma_x\sigma_y} \times \exp\left[-\frac{1}{2} \times \left(\frac{x-\mu_x}{\sigma_x}\right)^2\right] \times \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right] dy$$

$$= \left[\frac{1}{\sqrt{2\pi}\,\sigma_x} \exp\left(-\frac{u^2}{2}\right)\right] \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}v^2\right) dv\right]$$

(where $u = \frac{x-\mu_x}{\sigma_x}$, $v = \frac{y-\mu_y}{\sigma_y}$)

$$= \frac{1}{\sqrt{2\pi}\,\sigma_x} \exp\left[-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right] \times 1 \quad \text{and}$$

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i.e. $f_1(x) = \frac{1}{\sigma_x \sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right], -\infty < x < \infty, \sigma_x > 0$

and similarly,

$$f_2(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$
$$= \cdots$$
$$= \frac{1}{\sigma_y \sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right], \quad -\infty < y < \infty, \sigma_y > 0$$

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$\therefore \quad f(x,y) = f_1(x) f_2(y)$ if $\rho = 0$

$\Rightarrow$ X and Y are independent if $\rho = 0$

Conversely, let X and Y be independent r.v.'s.

It is known, if $(X, Y) \sim BVN(\mu_x, \mu_y, \sigma_x, \sigma_y, \rho)$ then $Cor(X, Y) = \rho$ (we shall prove it in the consequent lecture)

$\therefore \rho = 0$ //

Moment Generating Function (M.G.F.) of Bivariate normal variates (X, Y)

Theorem 1. Let $(X, Y) \sim BVN(\mu_x, \mu_y, \sigma_x, \sigma_y, \rho)$, then the MGF of (X, Y) is given by

$$M_X(t_1, t_2) = \exp\left[(t_1\mu_x + t_2\mu_y) + \frac{1}{2}(t_1^2\sigma_x^2 + t_2^2\sigma_y^2 + 2\rho t_1\sigma_x t_2\sigma_y)\right] \quad \text{--- (ii)}$$

Proof. By Definition,

$$M(t_1, t_2) = E\left[\exp(t_1 X + t_2 Y)\right]$$

$$= E\left[\exp\{t_1(\sigma_x U + \mu_x) + t_2(\sigma_y V + \mu_y)\}\right]$$

where $U = \dfrac{X - \mu_x}{\sigma_x}$, and $V = \dfrac{Y - \mu_y}{\sigma_y}$

$$= \frac{e^{t_1\mu_x + t_2\mu_y}}{2\pi\sqrt{1-\rho^2}} \cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{(t_1\sigma_x u + t_2\sigma_y v)} \times$$

$$\exp\left[-\frac{1}{2(1-\rho^2)}(u^2 - 2\rho uv + v^2)\right] du\, dv \quad \text{----(iii)}$$

Now, $u^2 - 2\rho uv + v^2 - 2(1-\rho^2)(t_1\sigma_x u + t_2\sigma_y v)$

$$= \left[(u-\rho v) - (1-\rho^2)t_1\sigma_x\right]^2 + \left[(1-\rho^2)(v - \rho t_1\sigma_x - t_2\sigma_y)^2 - t_1^2\sigma_x^2 - t_2^2\sigma_y^2 - 2\rho t_1 t_2\sigma_x\sigma_y\right] \quad \text{----(iv)}$$

Let $(u-\rho v) - (1-\rho^2)t_1\sigma_x = w\sqrt{1-\rho^2}$

and $v - \rho t_1\sigma_x - t_2\sigma_y = z$.

we get, $du\, dv = \sqrt{1-\rho^2}\, dw\, dz$ --(v)

Substituting from (iv) & (v) in (iii), we get

$$M(t_1, t_2) = \exp\left[(t_1\mu_x + t_2\mu_y) + \frac{1}{2}(t_1^2\sigma_x^2 + t_2^2\sigma_y^2 + 2\rho t_1 t_2\sigma_x\sigma_y)\right]$$

$$\times \left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{w^2}{2}} dw\right\}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz\right\}$$

$$\Rightarrow M(t_1, t_2) = \exp\left[(t_1\mu_x + t_2\mu_y) + \frac{1}{2}(t_1^2\sigma_x^2 + t_2^2\sigma_y^2 + 2\rho t_1 t_2\sigma_x\sigma_y)\right] \times 1 \times 1.$$

which is the result. //

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From the MGF of bivariate normal (ii), we can easily obtain the following:

$$E(X) = \frac{\partial}{\partial t_1} M(t_1, t_2)\Big|_{t_1=0}$$

$$= \mu_x ,$$

$$E(Y) = \frac{\partial}{\partial t_2} M(t_1, t_2)\Big|_{t_1=0}$$

$$= \mu_y ,$$

$$E(X^2) = \frac{\partial^2 M(t_1, t_2)}{\partial t_1^2}\Big|_{t_1=0} = \mu_x^2 + \sigma_x^2 ,$$

$$E(Y^2) = \frac{\partial^2}{\partial t_2^2} M(t_1, t_2)\Big|_{t_2=0} = \mu_y^2 + \sigma_y^2 ,$$

$$E(XY) = \frac{\partial^2}{\partial t_1 \partial t_2} M(t_1, t_2)\Big|_{t_1=t_2=0} = \rho\sigma_x\sigma_y + \mu_x\mu_y$$

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So, $Var(X) = \sigma_x^2$, $Var(Y) = \sigma_y^2$,

$Cov(X, Y) = \rho\sigma_x\sigma_y$, and

$$Cor(X, Y) = \frac{Cov(X, Y)}{\sigma_x\sigma_y} = \rho . \quad //$$

Note:-(1). The standardized bivariate normal distribution with parameter, ρ, is given by $BVN(0, 0, 1, 1, \rho)$ and its M.G.F. is given by

$$M(t_1, t_2) = \exp\left[\frac{1}{2}(t_1^2 + 2\rho t_1 t_2 + t_2^2)\right]$$

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(2). We have seen that if $(X, Y) \sim BVN(\mu_x, \mu_y, \sigma_x, \sigma_y, \rho)$ then the marginal density function of X and Y, are

$$f_1(x) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}$$

and $$f_2(y) = \frac{1}{\sigma_y \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_y}{\sigma_y}\right)^2}$$

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$\Rightarrow$ The marginal distribution of X and Y are, respectively $N(\mu_x, \sigma_x^2)$ and $N(\mu_y, \sigma_y^2)$

However the converse of this is not necessarily true,

that is, if the marginal of (X, Y) are normal, the joint distribution of (X, Y) may not be normal.

The converse is true if, and only if the linear combination $aX + bY$ is univariate normal for all a and b.

Theorem 2 If $(X, Y) \sim BVN(\mu_x, \mu_y, \sigma_x, \sigma_y, \rho)$, then the conditional density of X, given that $Y = y$, is the univariate normal density with parameters

$$\mu_x + \rho \frac{\sigma_x}{\sigma_y}(y - \mu_y) \quad \text{and} \quad \sigma_x^2(1-\rho^2).$$

Proof. We know

$$f_{1/2}(x/y) = \frac{f(x, y)}{f_2(y)}$$

$$\Rightarrow f_{1/2}(x/y) = \frac{1}{\sigma_x \sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)} \times \left\{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\left(1-(1-\rho^2)\right)\right\}\right]$$

$$= \frac{1}{\sigma_x \sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left[\frac{1}{2(1-\rho^2)\sigma_x^2}\left\{(x-\mu_x) - \rho\frac{\sigma_x}{\sigma_y}(y-\mu_y)\right\}^2\right]$$

$\Rightarrow$ $f_{1/2}(x/y)$ is the density function of a univariate normal distribution with parameters $\mu_x + \rho\frac{\sigma_x}{\sigma_y}(y-\mu_y)$ and $\sigma_x^2(1-\rho^2)$

i.e. $$f_{1/2}(x/y) \sim N\left(\mu_x + \rho\frac{\sigma_x}{\sigma_y}(y-\mu_y),\ \sigma_x^2(1-\rho^2)\right) \quad \text{--- (vi)}$$

One can show, similarly that,

$$f_{2/1}(y/x) \sim N\left(\mu_y + \rho\frac{\sigma_y}{\sigma_x}(x-\mu_x),\ \sigma_y^2(1-\rho^2)\right) \quad \text{--- (vii)}$$

Note: (1) The variance of $f_{2/1}(y/x)$ does not depend on x, and the variance of $f_{1/2}(x/y)$ is independent of y. This situation is known as '<u>Homoscedastic</u>' variance.

(2) The regression of Y on X, and that of X on Y are linear and given, respectively by

$$Y = \mu_y + \rho\frac{\sigma_y}{\sigma_x}(X-\mu_x), \text{ and}$$

$$X = \mu_x + \rho\frac{\sigma_x}{\sigma_y}(Y-\mu_y),$$

as $$E(Y/X=x) = \mu_y + \rho\frac{\sigma_y}{\sigma_x}(X-\mu_x), \text{ from (vii)}$$

and $$E(X|Y=y) = \mu_x + \rho\frac{\sigma_x}{\sigma_y}(Y-\mu_y), \text{ from (vi)}$$

(3) from (vi) & (vii), we get

$$\sigma_{X|y}^2 = \sigma_x^2(1-\rho^2),$$

and $$\sigma_{Y|x}^2 = \sigma_y^2(1-\rho^2)$$

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Example 26. Let $(X, Y) \sim BVN(0, 0, 2, \frac{\sqrt{17}}{3}, \rho)$, and $E(XY)=2$. Let $Z = 2X-3Y$. Determine

(a) the pdf of Z

(b) the conditional pdf of X, given $Z=z$,

(c) the joint pdf of X and Z.

Solution: Since $(X, Y) \sim BVN(0, 0, 2, \frac{\sqrt{17}}{3}, \rho)$, both X and Y represents univariate marginal distribution with pdf $f_x(x)$ and $f_y(y)$, respectively, i.e.

$X \sim N(0, 4)$ & $Y \sim N(0, \frac{17}{9})$

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(a) Also, a linear function of two jointly normal random variables is also normal. Thus, Z is normal, and

$$\sigma_Z^2 = E[(2X-3Y)^2] = 4E(X^2) + 9E(Y^2) - 12E(XY)$$
$$= 4(\sigma_x^2+\mu_x^2) + 9(\sigma_y^2+\mu_y^2) - 12\times 2$$
$$= 4(4+0) + 9\left(\tfrac{17}{9}+0\right) - 24$$
$$= 16 + 17 - 24$$
$$= 9$$

& $E(Z) = E(2X-3Y)$
$$= 2\mu_x - 3\mu_y = 0$$

$\Rightarrow Z \sim N(0, 9)$ with pdf given by

$$f_Z(z) = \frac{1}{\sqrt{2\pi}\cdot 3}\, e^{-\left(\frac{z^2}{18}\right)}, \quad -\infty < z < \infty$$

(b) Now, X and Y are linear functions of two independent normal random variables, so are X and Z, and

$$Cov(X, Z) = E(XZ) - \mu_x \mu_z$$
$$= E(XZ)$$
$$= E[X(2X-3Y)]$$
$$= 2E(X^2) - 3E(XY)$$
$$= 2\times 4 - 3\times 2 = 2$$

Next, $Cor(X, Z) = \rho_{XZ} = \frac{Cov(X, Z)}{\sigma_x \sigma_z} = \frac{2}{2\cdot 3} = \frac{1}{3}$

Thus the conditional expectation of X, given $Z = z$,

$$E(X|Z=z) = \mu_z + \rho_{XZ}\left(\frac{\sigma_x}{\sigma_z}\right) z = \rho_{XZ}\frac{\sigma_x}{\sigma_z} z$$
$$= \frac{1}{3}\times\frac{2}{3} z$$

$\Rightarrow \quad E(X/Z=z) = \frac{2}{9} z$

i.e. $E(X/Z) = \frac{2}{9} Z$, and

the conditional variance of X given $Z=z$, is

$$\sigma^2_{X|Z} = \sigma^2_X (1-\rho^2_{XZ})$$
$$= 4\left(1-\frac{1}{9}\right)$$
$$= \frac{32}{9}$$

$\Rightarrow \quad \sigma_{X|Z} = \frac{\sqrt{32}}{3}$.

Hence, the conditional pdf of X, given $Z=z$, is

$$f_{X|Z}(x/z) = \frac{3}{\sqrt{2\pi}\sqrt{32}}\, e^{-\frac{\left(x-\left(\frac{2z}{9}\right)\right)^2}{2\times 32/9}}$$

$$= \frac{3}{\sqrt{2\pi}\sqrt{32}}\, e^{-\frac{9}{64}\left(x-\frac{4z^2}{81}\right)^2}$$

$\left(\begin{array}{l} -\infty < x < \infty \\ -\infty < z < \infty \end{array}\right)$

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(c) Finally, the joint pdf of X and Z is

$$f_{X,Z}(x,z) = f_X(x) \times f_{X|Z}(x|z)$$

$$= \frac{1}{\sqrt{2\pi}\times 2\sqrt{1-\frac{1}{9}}}\, e^{-\frac{1}{2(1-1/9)}\times\frac{x^2}{4}} \times \frac{3}{\sqrt{2\pi}\sqrt{32}}\, e^{-\frac{9}{64}\left(x-\frac{4z^2}{81}\right)^2}$$

$$\Rightarrow f_{X,Z}(x,z) = \frac{1}{2\pi\sqrt{32}}\, e^{-\frac{\left[\frac{z^2}{9}+\frac{x^2}{4}-\frac{2}{3}\left(\frac{xz}{2\times 3}\right)\right]}{2(1-1/9)}}$$

$-\infty < x < \infty, \quad -\infty < z < \infty$

— .. —

Note that if X and Y are jointly normal, then each random variable X and Y is normal. However the converse is not true. Consider the following example.

Example 27. Let $X \sim N(0,1)$, and Y be independent of X, with $P(Y=1) = P(Y=-1) = \frac{1}{2}$. Let $Z = YX$

Show that,

(a) X and Z are uncorrelated, however they are dependent,

(b) X and Z are not jointly normal

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Solution (a) Clearly, Z is a normal variable with mean, $E(Z) = E(Y)\,E(X) = 0$.

Next, $E[XZ] = E[ZX^2] = E[Z]\,E[X^2] = 0 \times 1 = 0$

$\Rightarrow$ X and Z are uncorrelated, however, X and Z are clearly dependent.

(b) If X and Z were jointly normal, we would have a contradiction to our earlier conclusion that zero correlation implies independence. It follows that X and Z are not jointly normal even though both marginal distributions X and Z are normal. //