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Expectation of joint distribution

Let (X_1, X_2) be a random vector and let $Y = g(X_1, X_2)$ for some real-valued function g.

Suppose (X_1, X_2) is of continuous type.

The $E(Y)$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1, x_2)\, f_{X_1,X_2}(x_1, x_2)\, dx_1\, dx_2$$

if $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |g(x_1, x_2)|\, f_{X_1,X_2}(x_1, x_2)\, dx_1\, dx_2 < \infty$

In case, (X_1, X_2) is discrete,

$$E(Y) = \sum_{x_1}\sum_{x_2} g(x_1, x_2)\, p_{X_1,X_2}(x_1, x_2)$$

if $\displaystyle\sum_{x_1}\sum_{x_2} |g(x_1, x_2)|\, p_{X_1,X_2}(x_1, x_2) < \infty$

Now, we shall show that <u>E is a linear operator.</u>

Theorem. Let $Y_1 = g_1(X_1, X_2)$ and $Y_2 = g_2(X_1, X_2)$ be random variables such that $E(Y_1)$ and $E(Y_2)$ exist. Then, for all real numbers k_1 & k_2,

$$E(k_1 Y_1 + k_2 Y_2) = k_1 E(Y_1) + k_2 E(Y_2).$$

Proof: We shall first show that $E(k_1 Y_1 + k_2 Y_2)$ exist. We assume that (X_1, X_2) is of continuous type.

Consider $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |k_1 g_1(x_1, x_2) + k_2 g_2(x_1, x_2)|\, f_{X_1,X_2}(x_1, x_2)\, dx_1\, dx_2$

$$\leq |k_1| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |g_1(x_1, x_2)|\, f_{X_1,X_2}(x_1, x_2)\, dx_1\, dx_2$$

$$+ |k_2| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |g_2(x_1, x_2)|\, f_{X_1,X_2}(x_1, x_2)\, dx_1\, dx_2 \quad \text{---(i)}$$

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using the triangle inequality and linearity of integrals.

Since $E(Y_1)$ and $E(Y_2)$ exist,
both the integrals on the R.H.S. of (i) are finite ($< \infty$), and hence, the the integral on the L.H.S. of (i) is $< \infty$.

$\Rightarrow$ $E(k_1 Y_1 + k_2 Y_2)$ exist.

By Definition,

$$E(k_1 Y_1 + k_2 Y_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(k_1 g_1(x_1,x_2) + k_2 g_2(x_1,x_2)\right) \times f_{X_1,X_2}(x_1,x_2)\, dx_1\, dx_2$$

$$= k_1 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(x_1,x_2) f_{X_1,X_2}(x_1,x_2)\, dx_1\, dx_2$$

$$+ k_2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_2(x_1,x_2) f_{X_1,X_2}(x_1,x_2)\, dx_1\, dx_2$$

(by linearity of integrals).

$$= k_1 E(Y_1) + k_2 E(Y_2),$$

i.e. the desired result. //

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<u>Remark</u>. In case, we have a function $g(X_1)$ of X_1 (or $g(X_2)$ of X_2), the expected value of it can be found in two ways:

$$E(g(X_1)) = \begin{cases} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1) f_{X_1,X_2}(x_1,x_2)\, dx_1\, dx_2, & \text{(by def}^{n}.) \\ \int_{-\infty}^{\infty} g(x_1) f_{X_1}(x_1)\, dx_1 & \end{cases}$$

marginal pdf of X_1.

Consider the example :

<u>Example 1</u>, Let X_1 and X_2 have the pdf

$$f(x_1, x_2) = \begin{cases} 8x_1 x_2 & 0 < x_1 < x_2 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Evaluate $E(7X_1X_2^2 + 5X_2)$.

<u>Solution</u>: We know that E is a linear operator, hence

$$E(7X_1X_2^2 + 5X_2) = 7 \cdot E(X_1X_2^2) + 5\,E(X_2).$$

Now, by definition,

$$E(X_1X_2^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2^2 f(x_1, x_2)\,dx_1\,dx_2$$

$$= \int_0^1 \int_0^{x_2} 8x_1^2 x_2^3\,dx_1\,dx_2$$

$$= \int_0^1 \frac{8}{3} x_2^6\,dx_2 = \frac{8}{26} \quad \text{---(ii)}$$

and $$E(X_2) = \int_0^1 \left(\int_0^{x_2} x_2 (8x_1 x_2)\,dx_1 \right) dx_2 \quad (\text{by def}^n)$$

$$= \int_0^1 4x_2^4\,dx_2 = \frac{4}{5} \quad \text{---(iii)}$$

Thus, $E(7X_1X_2^2 + 5X_2) = 7 \times \frac{8}{21} + 5 \times \frac{4}{5} = \frac{20}{3}$ //.

<u>Note</u>: We observe that the marginal pdf of X_2,

$$f_2(x_2) = \int_0^{x_2} f(x_1, x_2)\,dx_1$$

$$= \begin{cases} \int_0^{x_2} 8x_1 x_2\,dx_1, & \text{if } 0 < x_2 < 1 \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} 4x_2^3 & \text{if } 0 < x_2 < 1 \\ 0 & \text{else} \end{cases}$$

$$\therefore E(X_2) = \int_{-\infty}^{\infty} x_2 f_2(x_2)\,dx_2 = \int_0^1 4x_2^4\,dx_2 = \frac{4}{5}, \text{ as (iii)} //$$

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Example 2: Let X_1, X_2 be two random variables with the joint pmf,

$$p(x_1, x_2) = \begin{cases} (x_1 + x_2)/12 & \text{for } x_1 = 1, 2, \; x_2 = 1, 2 \\ 0, & \text{elsewhere} \end{cases}$$

(i) Compute $E(X_1)$, $E(X_1^2)$, $E(X_2)$, $E(X_2^2)$, and $E(X_1X_2)$.

(ii) Is $E(X_1X_2) = E(X_1)E(X_2)$?

(iii) Find $E(2X_1 - 6X_2^2 + 7X_1X_2)$.

Solution: (i)

$$E(X_1) = \sum_{x_1=1}^{2}\sum_{x_2=1}^{2} x_1\, p(x_1, x_2)$$

$$= \sum_{x_1=1}^{2}\sum_{x_2=1}^{2} \frac{x_1(x_1+x_2)}{12}$$

$$= \sum_{x_1=1}^{2}\left[\frac{x_1(x_1+1)}{12} + \frac{x_1(x_1+2)}{12}\right]$$

$$= \left[\frac{1\cdot 2}{12} + \frac{1\cdot 3}{12}\right] + \left[\frac{2\cdot 3}{12} + \frac{2\cdot 4}{12}\right]$$

$$= \frac{5}{12} + \frac{14}{12} = \frac{19}{12},$$

$$E(X_1^2) = \sum_{x_1=1}^{2}\sum_{x_2=1}^{2} x_1^2\, p(x_1, x_2)$$

$$= \sum_{x_1=1}^{2}\sum_{x_2=1}^{2} \frac{x_1^2(x_1+x_2)}{12}$$

$$= \sum_{x_1=1}^{2}\left[\frac{(x_1^3 + x_1^2)}{12} + \frac{(x_1^3 + 2x_1^2)}{12}\right]$$

$$= \left[\frac{1^3 + 1^2}{12} + \frac{1^3 + 2\cdot 1^2}{12}\right] + \left[\frac{2^3 + 2^2}{12} + \frac{2^3 + 2\cdot 2^2}{12}\right]$$

$$= \left[\frac{2}{12} + \frac{3}{12}\right] + \left[\frac{12}{12} + \frac{16}{12}\right]$$

$$= \frac{5}{12} + \frac{28}{12} = \frac{33}{12} = \frac{11}{4}$$

Similarly, $E(X_2) = \sum_{x_1=1}^{2}\sum_{x_2=1}^{2} \frac{x_2(x_1+x_2)}{12} = \frac{19}{12}$, and

$E(X_2^2) = \frac{11}{4}$,

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(5)

and $E(X_1 X_2) = \sum_{x_1=1}^{2} \sum_{x_2=1}^{2} x_1 x_2 \frac{(x_1+x_2)}{12}$

$= \sum_{x_1=1}^{2} \left[\frac{x_1(x_1+1)}{12} + \frac{2x_1(x_1+2)}{12} \right]$

$= \left[\frac{2}{12} + \frac{2\cdot 3}{12}\right] + \left[\frac{2\cdot 3}{12} + \frac{4\cdot 4}{12}\right]$

$= \frac{8}{12} + \frac{22}{12} = \frac{30}{12} = \frac{5}{2}$

(ii) we have

$E(X_1)\,E(X_2) = \frac{19}{12} \times \frac{19}{12} = \frac{361}{144} \neq \frac{5}{2} = E\,X_1 X_2$

(iii) $E(2X_1 - 6X_2^2 + 7X_1X_2)$

$= 2\,E(X_1) - 6\,E(X_2^2) + 7\,E(X_1X_2)$, as E is a linear operator

$= 2 \times \frac{19}{12} - 6 \times \frac{11}{4} + 7 \times \frac{5}{2}$

$= \frac{19}{6} - \frac{33}{2} + \frac{35}{2} = \frac{19 - 99 + 105}{6} = \frac{25}{6}$ //

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Moment Generating function (MGF) of a Random Vector:

We define the MGF of a random vector $X = (X_1, X_2)'$, $M_{X_1,X_2}(t_1,t_2)$ by $M_{X_1,X_2}(t_1, t_2) = E(e^{t_1X_1+t_2X_2})$

if $E(e^{t_1X_1+t_2X_2})$ exists for $|t_1|<h_1$ and $|t_2|<h_2$,

where h_1 and h_2 are positive.

If $t=(t_1,t_2)'$, then we can write the MGF of X

as $M_{X_1,X_2}(t) = E[e^{t'X}]$

Note that, the mgfs of the marginal distributions of X_1 and X_2 are

$M_{X_1,X_2}(t_1, 0)$ and $M_{X_1,X_2}(0, t_2)$, respectively.

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Example 3. Let the continuous-type random variables X and Y have the joint pdf

$$f(x,y) = \begin{cases} e^{-x-y}, & 0<x<\infty,\ 0<y<\infty \\ 0, & \text{elsewhere} \end{cases}$$

Find the joint mgf of X and Y. Is $M_{X_1,X_2}(t_1,t_2) = M(t_1, 0)\,M(0, t_2)$?

Solution: By definition,

$$M_{X,Y}(t_1, t_2) = E(e^{t_1X+t_2Y})$$

$$= \int_0^\infty\int_0^\infty e^{t_1x+t_2y} f(x,y)\,dx\,dy$$

$$= \int_0^\infty\int_0^\infty e^{t_1x+t_2y}\, e^{-x-y}\,dx\,dy$$

$$= \int_0^\infty e^{(t_2-1)y}\left(\int_0^\infty e^{(t_1-1)x}dx\right)dy$$

$$= \int_0^\infty e^{(t_2-1)y} \times \frac{1}{(1-t_1)}\,dy \quad \text{if } t_1<1$$

$$= \frac{1}{(1-t_1)(1-t_2)} \quad \text{if } t_1<1 \text{ and } t_2<1$$

Next, the mgf's of the marginal distributions of X and Y are

$$
\begin{aligned}
M(t_1, 0) &= E\left(e^{t_1 X}\right) \\
&= \int_0^\infty \int_0^\infty e^{t_1 x} \cdot e^{-x-y} \, dx \, dy \\
&= \int_0^\infty e^{-y} \left(\int_0^\infty e^{-(1-t_1)x} \, dx \right) dy \\
&= \int_0^\infty e^{-y} \cdot \left(\frac{1}{1-t_1}\right) dy \quad \text{if } t_1 < 1 \\
&= \frac{1}{(1-t_1)} \quad \text{if } t_1 < 1
\end{aligned}
$$

and

$$
\begin{aligned}
M(0, t_2) &= E\left(e^{t_2 Y}\right) \\
&= \int_0^\infty \int_0^\infty e^{(t_2-1)y} \, e^{-x} \, dy \, dx \\
&= \frac{1}{(1-t_2)} \quad \text{if } t_2 < 1
\end{aligned}
$$

we have

$$
\begin{aligned}
M(t_1, 0)\, M(0, t_2) &= \frac{1}{(1-t_1)} \cdot \frac{1}{(1-t_2)} \quad \text{if } t_1 < 1 \text{ and } t_2 < 1. \\
&= M_{X,Y}(t_1, t_2),
\end{aligned}
$$

i.e. the result //

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Example 4. Let X_1, X_2 be two random variables with joint pmf

$$p(x_1, x_2) = \begin{cases} \left(\frac{1}{2}\right)^{x_1+x_2} & \text{for } 1 \le x_i < \infty,\ i=1,2 \\ 0 & \text{elsewhere} \end{cases}$$

where x_1 and x_2 are integers.

(i) Determine the joint mgf of X_1, X_2.

(ii) Show that $M(t_1, t_2) = M(t_1, 0)\,M(0, t_2)$.

Solution:

$$M_{X_1,X_2}(t_1,t_2) = E\left(e^{t_1X_1+t_2X_2}\right)$$

$$= \sum_{x_1=1}^{\infty}\sum_{x_2=1}^{\infty} e^{t_1x_1+t_2x_2}\left(\frac{1}{2}\right)^{x_1+x_2}$$

$$= \sum_{x_1=1}^{\infty}\left(e^{t_1}\frac{1}{2}\right)^{x_1}\sum_{x_2=1}^{\infty}\left(e^{t_2}\frac{1}{2}\right)^{x_2}$$

$$= \left[\frac{1}{1-2^{-1}e^{t_1}}-1\right]\left[\frac{1}{1-2^{-1}e^{t_2}}-1\right]$$

provided $t_i < \log 2$, $i = 1, 2$

Next, $$M(t_1, 0) = \sum_{x_1=1}^{\infty}\sum_{x_2=1}^{\infty} e^{t_1x_1}\left(\frac{1}{2}\right)^{x_1}\left(\frac{1}{2}\right)^{x_2}$$

$$= \left(\sum_{x_1=1}^{\infty}\left(e^{t_1}\frac{1}{2}\right)^{x_1}\right)\left(\sum_{x_2=1}^{\infty}\left(\frac{1}{2}\right)^{x_2}\right)$$

$$= \left[\frac{1}{1-2^{-1}e^{t_1}}-1\right]\times\left(\frac{\frac{1}{2}}{1-\frac{1}{2}}\right)$$

$$= \left[\frac{1}{1-2^{-1}e^{t_1}}-1\right], \text{ and (provided } t_1 < \log 2)$$

Similarly $M(0, t_2) = \left[\frac{1}{1-2^{-1}e^{t_2}}-1\right]$, provided $t_2 < \log 2$

Hence, $M(t_1, t_2) = M(t_1, 0)\,M(0, t_2)$, i.e. the result //